5-2016

Classification of Spacetimes with Symmetry

Jesse W. Hicks
Utah State University

CLASSIFICATION OF SPACETIMES WITH SYMMETRY

by

Jesse W. Hicks

A dissertation submitted in partial fulfillment
of the requirements for the degree

of

DOCTOR OF PHILOSOPHY

in

Mathematics

Approved:

Ian Anderson Major Professor	Zhaohu Nie Committee Member
Mark Fels Committee Member	Charles Torre Committee Member
Andreas Malmendier Committee Member	Mark R. McLellan, Ph.D. Vice President for Research and Dean of the School of Graduate Studies

UTAH STATE UNIVERSITY
Logan, Utah

2016

ABSTRACT

Classification of Spacetimes with Symmetry

by

Jesse W. Hicks, Doctor of Philosophy

Utah State University, 2016

Major Professor: Dr. Ian Anderson
Department: Mathematics and Statistics

Spacetimes with symmetry play a critical role in Einstein's Theory of General Relativity. Missing from the literature is a correct, usable, and computer accessible classification of such spacetimes. This dissertation fills this gap; specifically, we

i) give a new and different approach to the classification of spacetimes with symmetry using modern methods and tools such as the Schmidt method and computer algebra systems, resulting in ninety-two spacetimes;
ii) create digital databases of the classification for easy access and use for researchers;
iii) create software to classify any spacetime metric with symmetry against the new database;
iv) compare results of our classification with those of Petrov and find that Petrov missed six cases and incorrectly normalized a significant number of metrics;
v) classify spacetimes with symmetry in the book *Exact Solutions to Einstein's Field Equations Second Edition* by Stephani, Kramer, Macallum, Hoenselaers, and Herlt and in Komrakov's paper *Einstein-Maxwell equation on four-dimensional homogeneous spaces* using the new software.

(422 pages)

PUBLIC ABSTRACT

Classification of Spacetimes with Symmetry

Jesse W. Hicks

Spacetimes with symmetry play a critical role in Einstein's Theory of General Relativity. Missing from the literature is a correct, usable, and computer accessible classification of such spacetimes. This dissertation fills this gap; specifically, we

i) give a new and different approach to the classification of spacetimes with symmetry using modern methods and tools such as the Schmidt method and computer algebra systems, resulting in ninety-two spacetimes;

ii) create digital databases of the classification for easy access and use for researchers;

iii) create software to classify any spacetime metric with symmetry against the new database;

iv) compare results of our classification with those of Petrov and find that Petrov missed six cases and incorrectly normalized a significant number of metrics;

v) classify spacetimes with symmetry in the book *Exact Solutions to Einstein's Field Equations Second Edition* by Stephani, Kramer, Macallum, Hoenselaers, and Herlt and in Komrakov's paper *Einstein-Maxwell equation on four-dimensional homogeneous spaces* using the new software.

DEDICATION

All for you, Jodi.

ACKNOWLEDGMENTS

I would like to thank my advisor Dr. Ian Anderson for his tireless efforts in guiding me through the PhD and for being willing to accept me as his student. It is impossible to describe the outstanding nature of the rich learning experiences Dr. Anderson provided me with over the course of this research. For those experiences I'll always be grateful. I would also like to thank the members of the committee: Dr. Mark Fels for giving me my introduction to differential geometry and for offering sage advice through many thoughtful and helpful discussions over the years; Dr. Charles Torre for providing deep insight into many difficult concepts from general relativity and for always being willing to support my mathematical and professional missions; and Dr. Andreas Malmendier and Dr. Zhaohu Nie for their genuine concern for and aid toward my professional progress and for lending much needed advice and encouragement.

For making me dream bigger, work harder, and always push forward, I thank my wife Jodi.

Jesse W. Hicks

CONTENTS

LIST OF TABLES

CHAPTER 1

INTRODUCTION

1.1 Overview

Spacetimes are differentiable manifolds on which is defined a metric tensor of Lorentzian signature. The equivalence problem in general relativity is that of determining if two locally defined metric tensors are related by a coordinate transformation. While there is a theoretic solution, the implementation of this solution in a computer algebra system is very difficult (see Stephani, Kramer, Macallum, Hoenselaers, and Herlt [1], Section 9.2). The importance of classifying spacetimes in such a system cannot be overstated; there is a long history of "new" spacetimes being published only to later be retracted as the spacetime was already known. To make this problem more tractable, this dissertation investigates two restrictions of the equivalence problem, namely (*i*) the equivalence problem for spacetimes with symmetry and (*ii*) the equivalence problem for spacetimes found in the literature.

Four-dimensional spacetimes with symmetry play a central role in the theory of general relativity. In 1961, A.Z. Petrov published his work *Einstein Spaces* and gave a "complete" classification of four-dimensional spacetimes with symmetry according to local group action (see [2]). However, in its published form it is extremely difficult to use and its completeness and accuracy have been called into question, for example in Hicks [3], Bowers [4], and Fels [5]. The famous book *Exact Solutions of Einstein's Field Equations* [1] by Stephani et al. contains over 700 spacetimes and is a great place to begin solving the practical problem of determining if a solution is in the literature. The objectives of my research are the following:

1) Give a new and different approach to the classification of spacetimes with symmetry using modern methods and tools such as the Schmidt method (see Schmidt [6]) and computer algebra systems.
2) Create computer-based databases of the classification for easy access and use for researchers.
3) Create software to classify any spacetime metric with symmetry against the new database.
4) Classify spacetimes with symmetry in Stephani et al. [1] using the new software.

The results of this research are the following.

1) A complete list of all possible Lie algebra pairs $(\mathfrak{g}, \mathfrak{h})$ such that $3 \leq \dim(\mathfrak{g}) \leq 7$ and $\mathfrak{h}$ acts on $\mathfrak{g}/\mathfrak{h}$ as a subalgebra of $\mathfrak{so}(k, 1)$, for $k = 2, 3$, is given.
2) A database of Lie algebraic properties has been constructed which uniquely identifies each Lie algebra pair. Software was created which classifies Lie algebra pairs against the new database.
3) A complete list of locally defined Lorentz metrics with isometry dimension three through seven is given. These metrics are listed according to dimension and orbit type and include the cases of non-reductive isometry algebra. The metrics are normalized by the residual group. The result is a list of ninety-two inequivalent metrics. This together with the non-simple G spacetimes (30.8), (32.18), (32.20), (32.26), and (33.40)(C) of Petrov [2], completely solves the equivalence problem for Lorentzian metrics with isometry dimension three through seven.
4) We compare results of our classification with those of Petrov and find that Petrov missed six cases and incorrectly normalized a significant number of metrics.
5) The construction of metrics listed in 3) does not depend upon the imposition of the Einstein field equations. In this dissertation we relate known exact solutions to the Einstein field equations such as those in Stephani et al. [1] to the metrics of this new classification. We also give the symmetry classification of relevant Lie algebra pairs found in Komrakov [7].

The dissertation will be divided into the following chapters. Chapter 2 will contain mathematical preliminaries needed throughout the dissertation. Chapter 3 will implement and summarize the results of the Schmidt method, giving the complete classification of Lorentzian Lie algebra-subalgebra pairs. Chapter 4 describes software written for this dissertation to aid researchers wishing to classify spacetimes with symmetry. Chapter 5 will construct vector fields and invariant quadratic forms associated to the Lie algebra-subalgebra pairs of Chapter 3. Chapter 6 will provide commentary on Petrov's classification in [2], correcting all errors, and give the symmetry classification of each entry in [2] using the classification of this dissertation. Chapter 7 will give the classification of relevant pseudo-Riemannian pairs in Komrakov [7], wherein Komrakov gives a complete local classification of four-dimensional Einstein-Maxwell homogeneous spaces with an invariant pseudo-Riemannian metric of arbitrary signature. Chapter 8 gives the symmetry classification for exact solutions in Stephani et al. [1].

1.2 Chapter summaries

We briefly summarize the content of each chapter of the dissertation.

Chapter 2. Mathematical Preliminaries: We will cover notation, terminology, definitions, and theorems necessary for the later chapters.

Chapter 3. The Classification of Lorentz Lie algebra-subalgebra Pairs: A Lie algebra–subalgebra pair $(\mathfrak{g}, \mathfrak{h})$ is a Lie algebra $\mathfrak{g}$ with a chosen subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We say two Lie algebra-subalgebra pairs $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}', \mathfrak{h}')$ are equivalent if there is a Lie isomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ such that $\phi(\mathfrak{h}) = \mathfrak{h}'$. A Lie algebra-subalgebra pair $(\mathfrak{g}, \mathfrak{h})$ is *Lorentzian* if under the adjoint action $\mathfrak{h}$ acts as a subalgebra of the Lorentz algebra $\mathfrak{so}(k, 1)$ on $\mathfrak{g}/\mathfrak{h}$ and $\dim(\mathfrak{g}/\mathfrak{h}) \leq k + 1$, for $k = 2, 3$.

The Schmidt method is an algebraic method for the classification of homogeneous spaces with prescribed linear isotropy representation. A homogeneous space is a differentiable manifold M on which is defined a smooth transitive group action under a Lie group G. For any homogeneous space M, there exists a G-equivariant diffeomorphism $\psi : M \to G/H$, where H is the isotropy subgroup of G at any point of M. The quotient G/H is how we typically think of a homogeneous space. The Schmidt method, however, works at the algebraic level and we will employ the method to find Lorentzian Lie algebra-subalgebra pairs $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H respectively, noting we'll refer to $\mathfrak{h}$ as the *isotropy*. We shall find all possible non-equivalent Lorentzian Lie algebra-subalgebra pairs. It's worth noting the only possibilities for the dimensions of non-trivial Lorentzian Lie algebra-subalgebra pairs. To that end, recall that if G is the Lie group of isometries of a metric γ on a differentiable manifold M of dimension n, then

$$\dim G \leq \frac{n(n+1)}{2}.$$

Note in the case $M \equiv G/H$ is a homogeneous space, we have

$$4 \geq n = \dim\, M = \dim\, G - \dim\, H.$$

These facts combine to give the possibilities as $(\dim \mathfrak{g}, \dim \mathfrak{h}) = (3,1)$, $(4,1)$, $(5,1)$, $(5,2)$, $(6,2)$, $(6,3)$, $(7,3)$, for $\dim \mathfrak{g} \geq 3$. Note the cases $\dim\, G = 8, 9$ are excluded as the submaximal dimension is 7 (see Kobayashi [8]) and the cases $\dim\, G = 10$ are of constant curvature and are well understood (see Boothby [9], Section VIII.6). The cases of one or two dimensional group G will be included prior to publication of these results thereby completing the classification.

Technically the Schmidt method as implemented here will determine Lorentzian pairs where the isotropy is *reductive*, though the Schmidt method does not require reductive isotropy. The isotropy $\mathfrak{h}$ is reductive if there exists a subspace $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (vector space direct sum) and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. A pair is reductive if the subalgebra is reductive. The three and four dimensional reductive Lorentzian pairs with respective dimensions $(3,1)$ and $(4,1)$ were treated and given in Bowers [4]. The five-dimensional reductive Lorentzian pairs, $(5,1)$ and $(5,2)$, were treated and given in Rozum [10]. We use the Schmidt method to find all remaining reductive pairs, namely the six and seven dimensional cases $(6,2)$, $(6,3)$, $(7,3)$. All cases of non-reductive isotropy were studied in Fels [5], wherein is given all non-reductive Lorentzian Lie algebra-subalgebra pairs.

Combining the results of this dissertation with those before us, we summarize the complete classification of Lorentzian Lie algebra-subalgebra pairs in Appendix A.1.

Chapter 4. Software for Classification of Lorentzian Pairs: For any four-dimensional Lorentzian metric with symmetry, one would like to be able to compare the corresponding Lie algebra of Killing vectors with the Lorentzian pairs in our classification. In this chapter we will construct a database of all Lorentzian pairs from Chapter 3 and describe in detail new software for the classification of Lorentzian Lie algebra-subalgebra pairs. This software is the key tool that will be used in the subsequent sections of the dissertation.

The classifier takes as input a Lie algebra-subalgebra pair. It then generates a list of Lie-theoretic invariants for the abstract Lie algebra-subalgebra pair. As it computes these invariants, it compares them to an internal database of these same properties that have been pre-computed for the Lie algebra-subalgebra pairs of our new classification. From this comparison it's determined which entry in our database has the same properties in common with those supplied by the user to the program. Note that the database contains enough invariants such that its entries have been distinguished one from another. However, for proof of equivalence of pairs an explicit isomorphism ϕ is required. Software was created for this dissertation to find such an isomorphism and we will discuss it briefly.

Note that by adapting the software to accept a Lie algebra of Killing vectors Γ of a four-dimensional Lorentzian metric g and a point p, we easily classify the symmetry of g against the results of Chapter 3.

Chapter 5. Symmetries of Spacetimes: Here we pass from the algebraic to the geometric interpretation of our work. We will describe three main approaches to associating vector field systems Γ to the Lorentzian pairs of Chapter 3. The first approach is by constructing vector fields on the group G, then dropping them down to G/H under the projection map. The second approach is inductive in nature and is achieved by applying the first approach to solvable subalgebras then solving for remaining vector fields by imposing conditions determined by the bracket relations and the action of the isotropy. The third approach is using the book "Einstein Spaces" by A.Z. Petrov [2] and the software of Chapter 4 to determine which of Petrov's entries give vector field systems Γ whose real abstract Lie algebra-subalgebra pairs $(\mathfrak{g}, \mathfrak{h})$ are those discovered in Chapter 3.

For each Lie algebra of vector fields Γ we compute a basis $\mathfrak{G}$ of Γ-invariant symmetric rank-2 covariant tensors. Then Γ is a basis for all Killing vectors of the general invariant Lorentzian metric tensor g formed from $\mathfrak{G}$ since it was ensured in Chapter 3 that each Lorentzian pair was maximal (see Chapter 3 for the definition of maximal in this context). A process of normalization is then employed which may reduce the general metric g to an equivalent metric $\tilde{g}$. Normalize refers to eliminating extraneous parameters or functions showing up in the local coordinate expression of g. This process involves computing certain transformations called residual diffeomorphisms which pullback g all while keeping its Killing vectors undisturbed in their coordinate presentation. These residual diffeomorphisms are the flows of vector fields in the normalizer of Γ in the full infinitesimal pseudo-group of all vector fields on M.

The constructed Γ-invariant Lorentzian metrics g define spacetimes with symmetry and correspond to a unique Lorentzian Lie algebra-subalgebra pair of Chapter 3. We present the classification of (simple G) spacetimes with symmetry comprising the results of this chapter in Appendix A.2.

Chapter 6. The Petrov Classification: As mentioned above, in the book "Einstein Spaces" [2], Petrov claimed to have given a complete classification of spacetimes with symmetry. In this chapter we will

i) identify and correct typos and small errors in Petrov;
ii) identify Petrov entries for which the Killing vector field systems are diffeomorphic and give explicit diffeomorphisms;
iii) identify Petrov entries for which the given metric is not the most general invariant metric, allowing for proper normalization;
iv) identify Petrov entries for which the Killing vector field system for the given metric is larger than that provided by Petrov;
v) identify the non-reductive entries in Petrov;
vi) identify non-simple G Killing vector field systems in Petrov;
vii) give the symmetry classification of each simple G entry in Petrov using software from Chapter 4;
viii) identify the reductive Lorentzian pairs from 3 and the non-reductive from Fels [5] which do not appear in Petrov.

These steps comprise the bulk of efforts made to independently verify Petrov's results. However, to complete such a verification i) the normalization provided by Petrov for the G_4 on V_4 and G_3 on V_3 cases needs checking and ii) an independent classification of non-simple G spacetimes is needed.

Chapter 7. The Komrakov Classification of Einstein-Maxwell Spacetimes: In the paper *Einstein-Maxwell equation on four-dimensional homogeneous spaces* [7], Komrakov has given a classification of pseudo-Riemannian pairs. We identify and classify the subclass of Lorentzian pairs in [7] using the software of Chapter 4 and present the results.

Chapter 8. Symmetry Classification of Solutions to the Einstein Equations: The book *Exact Solutions of Einstein's Field Equations* by Stephani et al. [1] contains over 700 spacetimes. In this chapter we will use the software developed in Chapter 4 and the database created in Chapter 5 to give the symmetry classification of metrics in chapters 12 and 28 of Stephani et al. [1].

Chapter 9. Conclusion: We will briefly summarize the contributions of this work. The knowledge gained has opened many avenues for future research, for instance in the burgeoning field of five-dimensional relativity theory, or in classifying complex structures, symplectic geometries, Bach tensors, and so on.

Appendix A. Results: Here will be displayed tables of the Lorentzian Lie algebra-subalgebra pairs discovered in Chapter 3 including restrictions on any parameters, the isotropy subalgebra, and the isotropy type. A second section will present the results of Chapter 5, namely the classification of spacetimes with symmetry associated to the Lorentzian pairs.

Appendix B. Maple worksheets for the Schmidt Method: Here will be presented as verification the Maple worksheets showing the details of the Schmidt method for the $(6,2),(6,3),(7,3)$ cases of Lorentzian Lie algebra-subalgebra pairs.

CHAPTER 2

PRELIMINARIES

This chapter gives a brief review of concepts needed for understanding Lie algebras, manifolds, group actions, and spacetimes. An introduction to Lie theory can be found in Snobl [11]. More detailed discussions of manifolds and pseudo-Riemannian manifolds can be found in any introductory differential geometry text, for instance Boothby [9]. We will also introduce the Schmidt method (see Schmidt [6]) in this chapter. This method is employed to classify Lie algebra-subalgebra pairs in Chapter 3.

2.1 Lie Algebras

We review basic definitions of Lie algebras and related concepts. For a fuller discussion see any introductory text on Lie Theory. For further discussion of these topics with examples in Maple, see Hicks [3].

Definition 2.1.1. A *Lie algebra* $\mathfrak{g}$ is a vector space over a field $\mathbb{F}$ with a bilinear operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies

i) the Jacobi identity: $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$, for all $x, y, z \in \mathfrak{g}$,
ii) and the skew-symmetric property: $[x,y] = -[y,x]$, for all $x, y \in \mathfrak{g}$.

We call the product $[\cdot,\cdot]$ the Lie bracket. Given a basis $\{e_i\}$, note that $[e_i, e_j] = C_{ij}{}^k e_k$, for some $C_{ij}{}^k$, called the structure constants. Two Lie algebras are *isomorphic* if there exists a bijective linear transformation ϕ such that $\phi([x,y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}$.

A *subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ is a subspace closed under the Lie bracket of $\mathfrak{g}$. A Lie algebra-subalgebra pair is a pair $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra.

Definition 2.1.2. We say two Lie algebra-subalgebra pairs $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{g}', \mathfrak{h}')$ are equivalent if there is a Lie isomorphism $\phi : \mathfrak{g} \to \mathfrak{g}'$ such that $\phi(\mathfrak{h}) = \mathfrak{h}'$.

The study of spacetimes with symmetry undertaken in this dissertation is largely from an algebraic point of view. The following overview of Lie algebraic topics will be used to study the equivalence of Lie algebra-subalgebra pairs associated to such spacetimes.

Definition 2.1.3. An *ideal* of $\mathfrak{g}$ is a subspace $\mathfrak{a}$ of $\mathfrak{g}$ such that $[\mathfrak{a}, \mathfrak{g}] \subseteq \mathfrak{a}$, where $[\mathfrak{a}, \mathfrak{g}] = \text{span}\{[X, Y] \mid X \in \mathfrak{a}, Y \in \mathfrak{g}\}$.

Note that an ideal is also a Lie subalgebra since for any ideal $\mathfrak{a}$ in $\mathfrak{g}$ we clearly have $[\mathfrak{a}, \mathfrak{a}] \subseteq \mathfrak{a}$. Also observe that $\mathfrak{g}$ is an ideal of itself since it is closed under the Lie bracket. Due to the skew-symmetry of the Lie bracket there is no need to distinguish between left or right ideals.

Definition 2.1.4. A Lie algebra $\mathfrak{g}$ with only the trivial ideals $\{0\}$ and $\mathfrak{g}$ itself is called *simple*.

Definition 2.1.5.

i) Let $\mathfrak{h}_i$ be Lie algebras, for $i = 1, \ldots, n$. If $\mathfrak{g} = \{h_1 + h_2 + \cdots + h_n \mid h_i \in \mathfrak{h}_i\} = \sum \mathfrak{h}_i$ and $\mathfrak{h}_i \cap \mathfrak{h}_j = 0$ whenever $i \neq j$, then $\mathfrak{g}$ is said to be the external *direct sum* of the Lie algebras $\mathfrak{h}_i$, written $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \cdots \oplus \mathfrak{h}_n$. If each $\mathfrak{h}_i$ is a subalgebra, we say $\mathfrak{g}$ is the internal *direct sum* of the Lie subalgebras $\mathfrak{h}_i$, though in context it's typically clear and thus we often drop the qualifiers internal or external.

ii) A Lie algebra $\mathfrak{g}$ is said to be *indecomposable* if it cannot be written as a direct sum of Lie algebras and *decomposable* if it can.

iii) If $\mathfrak{h}$ and $\mathfrak{k}$ are Lie subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$ and $[\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{h}$, then $\mathfrak{g}$ is said to be the *semidirect sum* of $\mathfrak{h}$ and $\mathfrak{k}$. This will be denoted by $\mathfrak{g} = \mathfrak{h} \ni_s \mathfrak{k}$

Given a Lie algebra $\mathfrak{g}$, we are able to construct a Lie algebra of transformations in the following way.

Definition 2.1.6. Let $D : \mathfrak{g} \to \mathfrak{g}$ be a $\mathbb{F}$-linear transformation on a Lie algebra $\mathfrak{g}$. Then D is a *derivation* of $\mathfrak{g}$ if $D([x, y]) = [D(x), y] + [x, D(y)]$, for all $x, y \in \mathfrak{g}$. The set of derivations $\mathcal{D}(\mathfrak{g})$ of $\mathfrak{g}$ is easily seen to be a Lie algebra over $\mathbb{F}$ with Lie bracket given by the commutator $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$.

The Lie algebra $\mathcal{D}(\mathfrak{g})$ is typically larger than $\mathfrak{g}$ and its Lie theoretic invariants are useful in classifying Lie algebras as is done in Chapter 4.

Definition 2.1.7. The *derived algebra* is the ideal $\mathfrak{g}^{(1)} := [\mathfrak{g}, \mathfrak{g}]$.

If we define $\mathfrak{g}^{(0)} := \mathfrak{g}$, we can inductively define the chain of ideals

$$\mathfrak{g}^{(k)} := [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}]$$

for $k \geq 1$. Observe that as a consequence of this definition,

$$\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \cdots.$$

The sequence $\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \cdots$ will be called the *derived series*.

Definition 2.1.8. A Lie algebra $\mathfrak{g}$ is said to be *solvable* if for some k we have $\mathfrak{g}^{(k)} = 0$.

Definition 2.1.9. Let $\mathfrak{g}_{(0)} := \mathfrak{g}$. Then inductively define

$$\mathfrak{g}_{(i)} = [\mathfrak{g}, \mathfrak{g}_{(i-1)}].$$

The sequence $\mathfrak{g} = \mathfrak{g}_{(0)} \supseteq \mathfrak{g}_{(1)} \supseteq \mathfrak{g}_{(2)} \supseteq \cdots$ is called the *lower central series*.

Definition 2.1.10. If there exists a k such that $\mathfrak{g}_{(k)} = 0$, we say $\mathfrak{g}$ is *nilpotent*.

Proposition 2.1.11. Every nilpotent Lie algebra is solvable.

Proof. Let $\mathfrak{g}$ be a nilpotent Lie algebra. We'll proceed by way of induction. By definition of lower central series and derived series, we have $\mathfrak{g}^{(1)} = \mathfrak{g}_{(1)}$. Therefore $\mathfrak{g}^{(1)} \subseteq \mathfrak{g}_{(1)}$. Suppose $\mathfrak{g}^{(n)} \subseteq \mathfrak{g}_{(n)}$, for some $n \geq 1$, and let $x \in \mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}]$. Then $x = \sum c_i[x_i, y_i]$, for $x_i, y_i \in \mathfrak{g}$. Note that $x_i \in \mathfrak{g}$ and by the induction hypothesis $y_i \in \mathfrak{g}_{(n)}$. Thus $x = \sum c_i[x_i, y_i] \in [\mathfrak{g}, \mathfrak{g}_{(n)}] = \mathfrak{g}_{(n+1)}$, giving $\mathfrak{g}^{(n+1)} \subseteq \mathfrak{g}_{(n+1)}$. ∎

Definition 2.1.12. Let $\mathfrak{g}$ be a Lie algebra. The *radical* of $\mathfrak{g}$ is the maximal solvable ideal in $\mathfrak{g}$. We will denote the radical by $Rad(\mathfrak{g})$.

Definition 2.1.13. Let $\mathfrak{g}$ be a Lie algebra. The *nilradical* of $\mathfrak{g}$ is the maximal nilpotent ideal in $\mathfrak{g}$. We will denote the nilradical by $Nil(\mathfrak{g})$.

Note the radical (nilradical) is unique since the sum of solvable (nilpotent) ideals is a solvable (nilpotent) ideal. An immediate consequence of these definitions is that the radical of any abelian, nilpotent, or solvable Lie algebra $\mathfrak{g}$ is $\mathfrak{g}$ itself, namely $\mathrm{Rad}(\mathfrak{g}) = \mathfrak{g}$. And in any abelian or nilpotent Lie algebra $\mathfrak{g}$, we have $\mathrm{Nil}(\mathfrak{g}) = \mathfrak{g}$.

Definition 2.1.14. A Lie algebra $\mathfrak{g}$ is *semisimple* if $\mathrm{Rad}(\mathfrak{g}) = 0$.

Here is a powerful theorem with proof in Fulton and Harris [12], page 499.

Theorem 2.1.15. *(Levi's Decomposition Theorem)* Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g} = \mathrm{Rad}(\mathfrak{g}) \ni \mathfrak{s}$, where $\mathfrak{s}$ is a semisimple Lie subalgebra of $\mathfrak{g}$.

Definition 2.1.16. The decomposition of Theorem 2.1.15 will be called the *Levi decomposition*.

Definition 2.1.17. The *adjoint* of a vector x in a Lie algebra $\mathfrak{g}$ is a linear transformation $ad\,(x) : \mathfrak{g} \to \mathfrak{g}$ given by $ad\,(x)\,(y) = [x, y]$.

Definition 2.1.18. The *Killing form* K on a Lie algebra $\mathfrak{g}$ is the symmetric bilinear form given by $K\,(x, y) = \mathrm{tr}\,(ad\,(x)\,ad\,(y))$.

We state here without proof Cartan's criteria for semisimple and solvable Lie algebras (see [13] page 82).

Theorem 2.1.19. A Lie algebra is semisimple if and only if the Killing form is non-degenerate.

Definition 2.1.20. Let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. The *centralizer* $C_{\mathfrak{g}}\,(\mathfrak{h})$ is defined as

$$C_{\mathfrak{g}}\,(\mathfrak{h}) := \{x \in \mathfrak{g} \mid \forall y \in \mathfrak{h}, [x, y] = 0\}\,.$$

and the *normalizer* $\mathrm{Nor}_{\mathfrak{g}}\,(\mathfrak{h})$ is defined as

$$\mathrm{Nor}_{\mathfrak{g}}\,(\mathfrak{h}) := \{x \in \mathfrak{g} \mid \forall y \in \mathfrak{h}, [x, y] \in \mathfrak{h}\}\,.$$

The *generalized center* $GC_{\mathfrak{g}}(\mathfrak{h})$ is defined to be

$$GC_{\mathfrak{g}}\,(\mathfrak{h}) := \{x \in \mathfrak{g} \mid \forall y \in \mathfrak{g}, [x, y] \in \mathfrak{h}\}\,.$$

Definition 2.1.21. Let $Z^0\,(\mathfrak{g}) := C_{\mathfrak{g}}\,(\mathfrak{g}) \equiv GC_{\mathfrak{g}}\,(0)$. Then inductively define

$$Z^i(\mathfrak{g}) := GC_{\mathfrak{g}}(Z^{i-1}(\mathfrak{g})).$$

Observe that $Z^i(\mathfrak{g}) \subset Z^{i+1}(\mathfrak{g})$. We define the *upper central series* to be the chain of ideals

$$Z^0(\mathfrak{g}) \subseteq Z^1(\mathfrak{g}) \subseteq Z^2(\mathfrak{g}) \subseteq \cdots \subseteq \mathfrak{g}$$

Suppose $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra. If there exists a subspace $\mathfrak{m}$ such that $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ as a vector space direct sum and $[\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}$, then $\mathfrak{m}$ is a *reductive complement*. We then call $\mathfrak{m}$ a *symmetric complement* if $\mathfrak{m}$ additionally satisfies $[\mathfrak{m}, \mathfrak{m}] \in \mathfrak{h}$. We refer to the Lie algebra-subalgebra pair $(\mathfrak{g}, \mathfrak{h})$ as reductive or symmetric whenever such a complement $\mathfrak{m}$ to $\mathfrak{h}$ exists.

Given a Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$, the dual space $\mathfrak{g}^*$ to $\mathfrak{g}$ is defined to be the vector space of all linear functionals $\theta : \mathfrak{g} \to \mathbb{F}$. In a given basis $\{e_i\}$ with structure constants ${C_{ij}}^k$ with dual basis $\{\theta^i\}$, one may define the *exterior derivative* of θ^k by

$$d\theta^k \equiv -\frac{1}{2}{C_{ij}}^k \theta^i \wedge \theta^j. \tag{2.1}$$

This is called Cartan's formula. The statement $d^2 \equiv 0$ is equivalent to the Jacobi identity.

2.2 Manifolds and group actions

For a fuller discussion of topics covered in this section, please see Boothby [9]. Let M be a differentiable manifold and $p \in M$. Denote the set of all smooth real valued functions defined on an open neighborhood U of p by $C^\infty(p)$. Let T_pM denote the space of smooth *derivations* $X_p : C^\infty(p) \to \mathbb{R}$. The tangent bundle is defined as $T(M) = \cup_{p \in M} T_pM$ and the natural projection $\pi : T(M) \to M$ is given by $X_p \mapsto p$. The set of all smooth vector fields $X : M \to T(M)$, $p \mapsto X_p \in T_pM$, on an n-dimensional differentiable manifold M will be denoted $\mathfrak{X}(M)$. Note that $\mathfrak{X}(M)$ can be seen to be a vector space over $\mathbb{R}$.

We may introduce a product on $\mathfrak{X}(M)$. The proof of the following can be found in many introductory texts on differential geometry, for instance Boothby [9].

Proposition 2.2.1. The mapping $[\cdot, \cdot] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$, given by $[X, Y](f) = X(Y(f)) - Y(X(f))$, called the *commutator of vector fields*, defines a smooth vector field.

The vector space $\mathfrak{X}(M)$ is infinite-dimensional and a real Lie algebra under this commutator. We will see finite-dimensional Lie algebras of vector fields (as subalgebras of $\mathfrak{X}(M)$) later.

Definition 2.2.2. An *integral curve* of the vector field X on a manifold M is a smooth map $\tau : I \to M$, with I an open interval of $\mathbb{R}$, such that $\tau'(t) = X_{\tau(t)}$ for all $t \in I$. The integral curve is uniquely specified by the initial condition $\tau(0) = p$.

Definition 2.2.3. Let X be a smooth vector field on a manifold M. The *flow* of X is the one-parameter family of diffeomorphisms $\phi_t : M \to M$ where $t \in (-\epsilon, \epsilon)$ satisfying $\frac{d}{dt}\phi_t(x) = X_{\phi_t(x)}$ for all $x \in M$ in addition to $\phi_t \circ \phi_{t'} = \phi_{t+t'}$ for $t, t', t+t' \in (-\epsilon, \epsilon)$.

We need to define special types of mappings on manifolds.

Definition 2.2.4. Let $\phi : M \to N$ be a smooth map of manifolds.

i) The pushforward of $X_p \in T_pM$ under ϕ *at* p is the map $\phi_* : T_pM \to T_{\phi(p)}N$ given by $\phi_*(X_p)_{\phi(p)}(f) = X_p(f \circ \phi)$ for $f \in C^\infty(N)$.
ii) The pullback of $f : N \to \mathbb{R}$ under ϕ is the map $\phi^* : C^\infty(\phi(p)) \to C^\infty(p)$ defined by $\phi^*(f) = f \circ \phi$.
iii) The pullback of a cotangent vector $\theta : T_{\phi(p)}N \to \mathbb{R}$ under ϕ is the map $\phi^* : T^*_{\phi(p)}N \to T^*_pM$ given by $\phi^*(\theta)(X_p) = \theta(\phi_*(X_p))$.

Vector fields can be used to define directional rates of change of *tensors* on manifolds.

Definition 2.2.5. Let V be a vector space. An (r, s)-type tensor T on $V^s \times V^{*r} := V \times \cdots \times V \times V^* \times \cdots \times V^*$, for s copies of V and r copies of V^*, is a mapping $T : V^s \times V^{*r} \to \mathbb{R}$ in the set $\mathcal{T}^r_s$ of all $(s + r)$-linear maps (also denoted by $V^{*s} \otimes V^r$). A tensor is contravariant if $r \neq 0$ but $s = 0$ and covariant if $r = 0$ but $s \neq 0$.

The set $\mathcal{T}^r_s$ can be given the structure of a real vector space. There is much to be said regarding tensors and here the reader is only refreshed of the basic construct. Please see Boothby [9] for further study. However, note that for most applications in this dissertation, $V = T_pM$. Also, observe tangent vectors in T_pM are type $(1, 0)$ tensors and cotangent vectors in T^*_pM are type $(0, 1)$ tensors. An (r, s)-type tensor field on a differentiable manifold is the smooth assignment of an (r, s)-type tensor at each point of the manifold.

Note that for a diffeomorphism $\phi : M \to M$, we can extend to a map

$$\tilde{\phi} : \mathcal{T}^r_s \to \mathcal{T}^r_s. \tag{2.2}$$

As an illustration, if $T = X_1 \otimes \cdots \otimes X_r \otimes \lambda^1 \otimes \cdots \otimes \lambda^s \in \mathcal{T}^r_s$, then

$$\tilde{\phi}(T) = \phi_*(X_1) \otimes \cdots \otimes \phi_*(X_r) \otimes \phi^{-1^*}(\lambda^1) \otimes \cdots \otimes \phi^{-1^*}(\lambda^s),$$

where ϕ^{-1^*} is the pullback of the inverse of ϕ and where $\otimes$ denotes the tensor product given by $T \otimes S(v, w) = T(v)S(w)$ with $v, w \in V^s \times V^{*r}$ and $T, S \in \mathcal{T}^r_s$. This brings us to an important type of directional derivative.

Definition 2.2.6. The *Lie derivative* $\mathcal{L}_X T$ of a tensor field T along a vector field X at the point $p \in M$ is defined by

$$(\mathcal{L}_X T)_p = \frac{d}{dt}\Big|_{t=0} \left(\tilde{\phi}_t T\right)_p$$

where ϕ_t is the flow of X and $\tilde{\phi}$ the extended map in equation (2.2). The Lie derivative is an operation designed as a measure of the rate of change of a tensor T along the integral curve of a vector field X.

It can be shown that the following properties of the Lie derivative hold (see Stephani et al. [1] page 21):

- $\mathcal{L}_X f = X(f)$, for $f \in C^\infty(p)$.
- $\mathcal{L}_X(d\omega) = d(\mathcal{L}_X \omega)$, where d is the exterior derivative and ω a skew-symmetric $(0, s)$-type tensor.
- $\mathcal{L}_X(T \otimes S) = (\mathcal{L}_X T) \otimes S + T \otimes (\mathcal{L}_X S)$, for any tensors T and S, not necessarily of the same type.
- $\mathcal{L}_X Y = [X, Y]$, for vector field $Y \in \mathfrak{X}(M)$.

Next we define a special case of manifold having the additional structure of a group. Then we introduce group actions and relate them to vector fields.

Definition 2.2.7. Let G be a group. Then G is an m-dimensional Lie group if G is endowed with an m-dimensional C^∞ manifold structure with the additional properties

i. the group operation $* : G \times G \to G$, $(g, g') \mapsto g * g'$ is a C^∞ operation,

ii. the map $i : G \to G$ given by $i(g) = g^{-1}$ is C^∞.

Definition 2.2.8. Let G be any group and E any set. A left group action of G on E, written $G \circlearrowleft E$, is a mapping $\mu : G \times E \to E$ satisfying

i. $\mu(e, x) = x$, for all $x \in E$ and e the identity in G, and

ii. $\mu(g, \mu(g', x)) = \mu(gg', x)$, for all $g, g' \in G$, $x \in E$.

A similar definition can be made for a right action. We will say *"G acts on E by μ."*

We will be concerned with the case where $E = M$ is a differentiable manifold, G a Lie group, and μ is smooth and refer to μ as a *smooth action* or C^∞ *action.*

Remark 2.2.9. Fix $g \in G$. Then define $\mu_g : M \to M$ to be given by $\mu_g(x) = \mu(g, x)$. Observe that

$$\mu_g \circ \mu_h(x) = \mu_g(\mu_h(x)) = \mu_g(\mu(h, x)) = \mu(g, \mu(h, x)) = \mu(gh, x) = \mu_{gh}(x).$$

Then for $h = g^{-1}$, we have $\mu_g \circ \mu_{g^{-1}} = \mu_e$ and therefore $\mu_g \circ \mu_{g^{-1}} = \mathbb{1}_M$. Then $\mu_{g^{-1}} = \mu_g^{-1}$ and since μ is smooth, μ_g is smooth for all $g \in G$. Therefore μ_g is a diffeomorphism from M to M.

Note that a special case of a smooth group action is the flow $\psi : \mathbb{R} \times M \to M$ of a complete vector field X (we say X is complete if its flow curves are defined for all $t \in \mathbb{R}$). Observe that if we fix $t \in \mathbb{R}$, then ψ_t is a diffeomorphism and $\psi_t \circ \psi_s = \psi_{t+s}$ by Remark 2.2.9.

Definition 2.2.10. The *infinitesimal generator* for a flow ψ is the smooth vector field X on M given by $p \mapsto X_p$, where $X_p : C^\infty(p) \to \mathbb{R}$ is given by

$$X_p(f) = \lim_{t \to 0} \frac{f(\psi_t(p)) - f(p)}{t}.$$

For a proof that X so defined is a vector field, see Hicks [3]. Observe the following connection between smooth vector fields and flows (for proof, see Boothby [9]):

Theorem 2.2.11. Let X be a smooth vector field on a differentiable manifold M. Then X is the infinitesimal generator of a unique local flow ψ on M.

Note by *local* it's meant that for any given point $p \in \mathbb{R} \times M$, the action of the flow is defined in a neighborhood $W \subset \mathbb{R} \times M$ such that $p \in W$.

Definition 2.2.12. Let G be an n-dimensional Lie group, with identity e, acting on a differentiable manifold M by the C^∞ action μ. Let $\sigma : \mathbb{R} \to G$ be a 1-parameter subgroup of G. That is, $\sigma(0) = e$ and $\sigma(t+s) = \sigma(t)\sigma(s)$. Then the associated *infinitesimal generator* on M is the vector field defined by the flow $\psi_t(p) := \mu(\sigma(t), p)$.

Observe that

$$\psi_{t+s}(p) = \mu(\sigma(t+s), p) = \mu(\sigma(t)\sigma(s), p) = \mu(\sigma(t), \mu(\sigma(s), p)) = (\psi_t \circ \psi_s)(p)$$

and since μ is smooth, ψ_t is smooth.

2.3 Simple G spaces

We will now transition to Lie algebras naturally associated to Lie groups and give a widely known theorem regarding the structure of Lie subgroups. Then we will describe homogeneous spaces followed by simple G spaces. First we need more terminology.

Definition 2.3.1. Let G be a Lie group acting on a manifold M by a smooth action μ. The *orbit* of $x \in M$ under the action μ is the set $\mathcal{O}_G(x) = \{\mu(g, x) \mid g \in G\}$.

Definition 2.3.2. Let G act on M by μ. The *isotropy* at $x \in M$ is the subgroup

$$G_x = \{g \in G : \mu(g, x) = x\}.$$

The *global isotropy* is defined as the normal subgroup $N \equiv \bigcap_{x \in M} G_x$. The action μ is said to be *faithful* if N is trivial.

Definition 2.3.3. Suppose $G \circlearrowleft M$ by the smooth action μ. The *linear isotropy representation* of G_x at $x \in M$ is the group homomorphism $\rho_x : G_x \to GL(T_x M)$ given by $\rho_x(g)(X) = \mu_{g*}(X)$. The representation is *faithful* if for any X such that $\rho_x(g)(X) = X$ it follows that g is the identity in G.

Definition 2.3.4. Let G act on M by μ. If for any $x, y \in M$, there exists a $g \in G$ such that $\mu(g, x) = y$, then μ is said to be *transitive*. If μ is transitive, we say M is *homogeneous* under the action of G by μ.

Definition 2.3.5. Let G be Lie group with Lie subgroup H. The quotient of G by H is the set of left cosets $G/H := \{gH \mid g \in G\}$, where $gH = \{gh \mid h \in H\}$. The equality $g_1H = g_2H$ holds when $g_1h = g_2$ for some $h \in H$.

The Lie group G acts in a natural and transitive way on the quotient G/H by $g(hH) = (gh)H$ for $g, h \in G$. However, this action may not in general be faithful. Indeed, if $K \subset H \subset G$ is normal in G, then for $g \in G$ and $k \in K$ we have that $g^{-1}kg = \tilde{k}$ for some $\tilde{k} \in K$. But then $kg = g\tilde{k}$ and therefore $(kg)H = (g\tilde{k})H = g(\tilde{k}H) = gH$. Thus $k \in N$ for all $k \in K$ where N is the global isotropy given by the action $G \circlearrowleft G/H$. As an important side note, if given a smooth group action under the Lie group G, since the global isotropy N is a normal subgroup of G, it is always possible to replace the action of G by the *faithful* action of G/N

It can be shown that every finite-dimensional Lie group G has a corresponding Lie algebra $\mathfrak{g}$. Specifically, the left–invariant vector fields on G are the vector fields X such that $X_{gh} = L_{g^*}(X_h)$ for every $g, h \in G$, where L is the smooth group action $L : G \times G \to G$ given by $L(g, h) = gh$ (group multiplication on the left by g). The right–invariant vector fields are defined similarly. The set of left–invariant vector fields forms a Lie algebra on G under the commutator of vector fields. Conversely, given a finite-dimensional Lie algebra $\mathfrak{g}$, there always exists a simply-connected Lie group G whose Lie algebra of left-invariant vector fields is isomorphic to $\mathfrak{g}$ (see Warner [14] page 101). The following two theorems have proof in Boothby [9].

Theorem 2.3.6. (Closed Subgroup Theorem): Let G be a Lie group with H a closed subgroup under the subspace topology. Then H is an embedded Lie subgroup of G.

Corollary 2.3.7. Let G be a Lie group with closed subgroup H.

i) The canonical projection $\pi : G \to G/H$ is then smooth and induces the structure of a differentiable manifold on G/H.

ii) For each $gH \in G/H$ there exists a neighborhood U of gH and a smooth local cross-section $\sigma : U \to G$ such that $\pi \circ \sigma = \mathbb{1}_U$, the identity on $U \subset G/H$.

iii) G/H is a homogeneous space under the natural action of G.

Examples of the Closed Group Theorem and its Corollary 2.3.7 are found in Chapter 5. In this dissertation we are principally concerned with the structure of G/H when G is a Lie group acting on a manifold and H is the isotropy at a point.

Theorem 2.3.8. Suppose G is a Lie group acting smoothly on a manifold M by μ. Choose $x_0 \in M$. The isotropy G_{x_0} is closed in G as a topological subspace.

Proof. Note the isotropy group G_{x_0} is given by $G_{x_0} = \mu_{x_0}^{-1}(x_0)$. As the inverse image of a point under a smooth map, G_{x_0} is topologically closed. ∎

Definition 2.3.9. Suppose the Lie group G acts by μ and ν on manifolds M and N respectively. We say the map $\phi : M \to N$ is *G-equivariant* if $\phi(\mu(g,x)) = \nu(g, \phi(x))$ for every $g \in G$.

The following theorem, The Fundamental Theorem of Homogeneous Spaces, has proof in Boothby [9].

Theorem 2.3.10. (The Fundamental Theorem of Homogeneous Spaces): Let the Lie group G act smoothly on M by a transitive group action μ and for $x \in M$ let G act on G/G_x by group multiplication. Then there exists a G-equivariant diffeomorphism $\phi : M \to G/G_x$.

The theorem may be extended when the structure of the isotropy is in some regard independent of the point of reference. To make this idea more precise, note the following definition of a slice of a manifold at a point.

Definition 2.3.11. Suppose G is a Lie group acting on the manifold M by the smooth action μ. A *local cross-section*, S, is a submanifold of M such that for any $x \in S$ we have $T_xO_G(x) \oplus T_xS = T_xM$. If for each choice of $x_0 \in S$ there is a smooth function $\gamma : S \to G$ such that $\mu(\gamma(x_0), x_0) = x_0$ and $G_{x_0} = \gamma(y) G_y (\gamma(y))^{-1}$ for every $y \in S$, then we say S is a *local slice* of M and M is a *simple G space*. If $\gamma(S) \subset G_{x_0}$, and therefore $G_{x_0} = G_y$, then we say S is *isotropy preserving* (see Rozum [10] page 18).

Proposition 2.3.12. Suppose the Lie group G acts on the manifold M by the smooth action μ. If M admits a local slice S, then through an arbitrary point $x_0 \in S$, M admits a local isotropy preserving slice S'.

Proof. Choose $x_0 \in S$. By hypothesis there exists a $\gamma : S \to G$ such that $\mu(\gamma(x_0), x_0) = x_0$ and $G_{x_0} = \gamma(y) G_y (\gamma(y))^{-1}$ for every $y \in S$. Note the isotropy at any $y \in S$ has the form $G_y = (\gamma(y))^{-1} G_{x_0} \gamma(y)$. Then $S' := \{\mu(\gamma(s), s) \mid s \in S\}$ is a local slice with isotropy at each point given by G_{x_0} since both μ and γ are smooth. ∎

For proof of the following theorem and corollary, see Rozum [10], page 19.

Theorem 2.3.13. (The Fundamental Theorem of Simple G Spaces): Let the Lie group G act smoothly on the manifold M. Suppose there exists a local slice S such that the isotropy at points in S is the subgroup H. Then for any fixed $x \in S$, there exists a local G-equivariant diffeomorphism from a neighborhood U of x in M to $(S \cap U) \times G/H$.

This leads to the following result regarding the point-independence of isotropy in manifolds with group actions admitting slices.

Corollary 2.3.14. Suppose the Lie group G acts smoothly on the manifold M. If M admits a local slice S through $x \in M$, and the isotropy at each point in S is given by the subgroup H, then there exists a neighborhood U of x such that the isotropy at points in U conjugate to H.

We will relate these ideas to how they are used in this dissertation and give an example of their use in the next section.

2.4 Infinitesimal group actions

We now describe infinitesimal group actions. We begin with rudimentary concepts.

Definition 2.4.1. A finite-dimensional real Lie algebra of vector fields Γ on a manifold M is an *infinitesimal group action*.

Let Γ be an infinitesimal group action on $\mathcal{M}$.

Definition 2.4.2. The *isotropy subalgebra* of Γ at $p \in M$ is $\Gamma_p = \{X \in \Gamma \mid X_p = 0\}$.

Definition 2.4.3. The *linear isotropy representation* of Γ_p at $p \in M$ is the map $\rho_p : \Gamma_p \to \mathfrak{gl}(T_pM)$ given by $\rho_p(X)(Y) = [X, Y]_p$. As X vanishes at p, the derivatives of Y at p need not be known. The representation is *faithful* if $X = 0$ whenever $\rho_p(X)(Y) = 0$.

Definition 2.4.4. If $\{X_1(p), \ldots, X_r(p)\}$ spans T_pM for each $p \in M$, where $\Gamma = \{X_1, \ldots, X_r\}$, then the infinitesimal group action is *transitive*. In this case M is said to be *homogeneous* under the infinitesimal action of Γ.

Any smooth group action of a Lie group G on a manifold M generates an infinitesimal group action in the following way. Suppose the group action is given by μ. Let X be a left-invariant vector field on G with integral curve $\phi : I \to G$, where $(-\epsilon, \epsilon) \subset \mathbb{R}$. Then $\phi(0) = e$, the identity element in G. Note that $\frac{\mathrm{d}}{\mathrm{d}t}\mu(\phi(t), x)|_{t=0}$ is a tangent vector $Y_x \in T_xM$. Then letting x vary produces a vector field Y on M. This generates the map $\tau : T_eG \to \mathfrak{X}(M)$ whose image is an infinitesimal group action Γ on M. Furthermore, if $X(e) = X_e \in T_eG_x$, the tangent space to G_x at the identity, then $\phi(I) \subset G_x$ and therefore $\mu(\phi, x) = x$ and the tangent vector here must be the zero vector. Therefore $\tau(T_eG_x) = \Gamma_x$.

Now suppose $\Gamma = \{X_1, \ldots, X_s\}$ is an infinitesimal group action of orbit dimension r on a manifold M of dimension n. We can introduce coordinates x^a, y^β on M such that $a = 1, \ldots, r$ and $\beta = r+1, \ldots, n$ and $X_i = A_i^a(x,y)\frac{\partial}{\partial x^a}$ for $i = 1, \ldots, s$. We say the action is *simple* if there exists coordinates $\tilde{x}^a, \tilde{y}^\beta$ such that $X_i = B_i^a(\tilde{x})\frac{\partial}{\partial \tilde{x}^a}$. In the case of a simple action we have $[X_i, \partial_{\tilde{y}^\beta}] = 0$ and thus one easily proves the following theorem.

Theorem 2.4.5. The infinitesimal action of a Lie algebra of vector fields with r-dimensional orbits is simple if and only if there exist $n-r$ commuting vector fields Z_γ such that $[X_i, Z_\gamma] = 0$.

Local isotropy preserving slices may be studied at the infinitesimal level. Note the following example from Petrov [2].

Example 2.4.6. Consider the infinitesimal group action (32.26) in [2]

$$X_i = \partial_{x^i} \ (i = 1,2,3) \quad X_4 = x^2\partial_{x^1} + \omega\left(x^4\right)\partial_{x^2} + \lambda\left(x^4\right)\partial_{x^3}$$

with $(\frac{d\omega}{dx^4})^2 + (\frac{d\lambda}{dx^4})^2 \neq 0$. These are the Killing vectors of a family of metric tensors (see Section 2.5). The only non-zero commutator is $[X_2, X_4] = X_1$ and the isotropy at (a,b,c,d) is spanned by $h = bX_1 + \omega(d)X_2 + \lambda(d)X_3 - X_4$. If Γ is simple, then there exists a vector field Z satisfying Theorem 2.4.5. Since Z commutes with X_1, X_2, and X_3, we can write

$Z = A^i(x^4)\frac{\partial}{\partial x^i}$, $i = 1, \ldots, 4$. Then we must have

$$[X_4, Z] = -A^2\left(x^4\right)\partial_{x^1} + A^4\left(x4\right)\frac{\mathrm{d}}{\mathrm{d}x^4}\omega\left(x^4\right)\partial_{x^2} - A^4\left(x^4\right)\frac{\mathrm{d}}{\mathrm{d}x^4}\lambda\left(x^4\right)\partial_{x^3} = 0.$$

This is true if and only if $A^2(x^4) = A^4(x^4) = 0$ and consequently $Z = A^1(x^4)\frac{\partial}{\partial x^1} + A^3(x^4)\frac{\partial}{\partial x^3}$. However, Z is thus in the span of the orbits. We conclude that there is no vector field transverse to the orbits and therefore Γ is not simple.

2.5 Spacetimes

We wish to now settle the preceding group and algebraic work within a broader geometric province.

Definition 2.5.1.

i) A *metric tensor* g is a mapping on M given by $p \mapsto g_p$, where $g_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate symmetric bilinear form, and where g varies smoothly with $p \in M$.

ii) The matrix representation (g_{ij}) of $g(p)$ is determined by computing

$$g_{ij} = g(p)(X_i(p), X_j(p),$$

for a basis $\{X_i(p)\}$ of T_pM.

iii) We define a *quadratic form* Q_p on T_pM associated to g at p by $Q(X_p) = g_p(X_p, X_p)$.

iv) We may diagonalize the symmetric matrix (g_{ij}), where the diagonal values a_{ii} are in the set $\{0, -1, 1\}$. Note $a_{ii} \neq 0$ in our case as (g_{ij}) is non-degenerate. If r is the number of positive 1's in the diagonal and s the number of negative 1's, the *signature* of Q_p is defined to be (r, s). By Sylvester's Law of Inertia, the signature of Q_p is independent of how (g_{ij}) was diagonalized (see Greub [15], page 252).

v) If the signature (r, s) of Q_p is constant with respect to p, then the *signature of* g is (r, s).

vi) A *pseudo-Riemannian* n-manifold is one on which is defined a metric tensor with signature (r, s).

vii) A *Lorentzian* n-manifold or *spacetime* is one on which is defined a metric tensor having signature $(n-1, 1)$ or $(1, n-1)$. Typically $n = 4$ in the case of a spacetime.

viii) A *Riemannian* n-manifold is one on which is defined a metric tensor with signature $(r, 0)$, that is, a positive definite metric tensor.

Definition 2.5.2. Let (M, g) be a spacetime. The set of all diffeomorphisms $\phi : M \to M$ such that $g(X, Y) = g(\phi_* X, \phi_* Y)$ forms a Lie group under composition and is called the *isometry group*. Such a map ϕ is said to preserve g and is called an *isometry*.

Definition 2.5.3. The *isometry algebra* of a pseudo-Riemannian manifold (M, g) is the set Γ of all vector fields on M such that $\Gamma = \{X \in TM : \mathcal{L}_X g = 0\}$. If $X \in \Gamma$, then X is called a *Killing vector*.

The flow of a Killing vector X consists of local isometries. Note Γ is a finite-dimensional real Lie algebra under the commutator of vector fields. Therefore, Γ is an infinitesimal group action on M. For proof of the following theorem, see Kobayashi [8].

Theorem 2.5.4. The isometry group G of an n-dimensional spacetime is of dimension at most $\frac{n(n+1)}{2}$ and the corresponding Lie algebra $\mathfrak{g}$ is isomorphic to the isometry algebra Γ.

Definition 2.5.5. The *isotropy subalgebra* Γ_p at p in the pseudo-Riemannian manifold (M, g) is the subalgebra of the isometry algebra given by the vector fields that vanish at p.

For further details on the next two theorems, see Stephani et al. [1].

Theorem 2.5.6. If G is the isometry group of a pseudo-Riemannian manifold, then the isotropy subalgebra at p is the Lie algebra of the isotropy subgroup G_p.

Theorem 2.5.7. For any p in a pseudo-Riemannian manifold (M, g), the linear isotropy representation for the isotropy subalgebra Γ_p is faithful.

Theorem 2.5.8. The isotropy subgroup of a pseudo-Riemannian manifold (M, g) with signature (r, s) is isomorphic to a subgroup of $O(r, s)$.

Proof. Let G_p be the isotropy subgroup of the isometry group with Γ_p the isotropy subalgebra at p. By definition, for any $\phi \in G_p$ we have

$$g_{\phi(p)}(\phi_* X_p, \phi_* Y_p) = g_p(X_p, Y_p)$$

for $X_p, Y_p \in T_p M$. As $g_{\phi(p)} = g_p$, $\phi \in G_p$, the isotropy group preserves g_p, a quadratic form of signature (r, s). Therefore $G_p \subset O(s, r)$. ∎

The orbits at a point under the action of the isometry group in a spacetime are categorized in one of three categories according to the signature of the metric on the orbit. This finds application in Chapter 5.

Definition 2.5.9. Let V_k be a k-dimensional subspace of an n-dimensional spacetime such that the metric on V_k has constant signature. The *subspace type* of V_k is one of the following:

i) The subspace type is *spacelike* if the signature of the metric on the subspace is $(k, 0)$.
ii) The subspace type is *timelike* if the signature of the metric on the subspace is $(k-1, 1)$.
iii) The subspace type is *null* if the metric is degenerate on the subspace.

The *orbit type* of an orbit through a point is its subspace type.

Before we give the next theorem, we review a couple of facts. Recall for each $x \in \mathfrak{g}$ the adjoint derivation $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ given by $\mathrm{ad}(x)(y) = [x, y]$. We can "drop" this map in the natural way to the quotient $\mathfrak{g}/\mathfrak{h}$ by $\mathrm{ad}(x)_{\mathfrak{g}/\mathfrak{h}}(y + \mathfrak{h}) := \mathrm{ad}(x)(y) + \mathfrak{h}$. We say an inner product η (not necessarily positive-definite) on $\mathfrak{g}/\mathfrak{h}$ is $\mathrm{ad}(\mathfrak{h})$-invariant if $A^T\eta + \eta A = 0$ for all $x \in \mathfrak{h}$, where A is the matrix representation of $\mathrm{ad}(x)_{\mathfrak{g}/\mathfrak{h}}$. For proof of the following very important theorem see Kobayashi2 [16], Proposition 3.1.

Theorem 2.5.10. Suppose H is a connected subgroup of the Lie group G. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively. The G-invariant metrics on the quotient G/H are in one-to-one correspondence with $\mathrm{ad}(\mathfrak{h})$-invariant inner products on $\mathfrak{g}/\mathfrak{h}$.

As an example of this theorem, see Chapter 5, Section 5.2. Note that it can be shown when $\mathfrak{h}$ admits a reductive complement $\mathfrak{m}$, $\mathrm{ad}(\mathfrak{h})$-invariant inner products on $\mathfrak{g}/\mathfrak{h}$ correspond to (similarly defined) $\mathrm{ad}(\mathfrak{h})$-invariant inner products on $\mathfrak{m}$.

2.6 The isometry algebra of a metric

As we noted earlier, Killing vectors Γ of a metric always define a finite-dimensional Lie algebra which we call the isometry algebra. Somewhat surprisingly, it is possible to determine the dimension of the isometry algebra and even the structure constants of the algebra without explicitly knowing the Killing vector fields. From the Killing equation $\mathcal{L}_X g = \nabla_a X_b + \nabla_b X_a = 0$, where $X_a = g_{ab} X^b$, one can show $\nabla_a \nabla_b X_c = X_d R^d{}_{abc}$, with

$R^d{}_{abc}$ the components of the curvature tensor of g. Therefore all derivatives of X^a at a point p are determined by the pairs $(X^a, \nabla_b X^c)$, known as the *Killing data*. The set of Killing data is a vector space $\mathcal{K}$ with $\dim(\mathcal{K}) \leq \frac{n(n+1)}{2}$.

It follows from the Lie derivative identity

$$\mathcal{L}_X T^{a\ldots}_{b\ldots} = X^c \nabla_c T^{a\ldots}_{b\ldots} + (\nabla_b X^c) T^{a\ldots}_{c\ldots} - (\nabla_c X^a) T^{c\ldots}_{b\ldots} + \cdots$$

that the Killing data satisfy the system of linear equations

$$E_0 \equiv \mathcal{L}_X R^a{}_{bcd} = 0,\ E_1 \equiv \mathcal{L}_X (\nabla_m R^a{}_{bcd}) = 0,\ \ldots,\ E_k \equiv \mathcal{L}_X (\nabla_{m_1} \nabla_{m_2} \cdots \nabla_{m_k} R^a{}_{bcd}) = 0.$$

Let r_k be the rank of the system $(E_0,\ \ldots,\ E_k)$. In the analytic setting, if $r_{k+1} = r_k$, then $r_{k+j} = r_k$ for all $j > 1$. Furthermore, the vector space of Killing vectors has dimension $\frac{n(n+1)}{2} - r_k$ in a suitable neighborhood of a point p (see Petrov [2] and Eisenhart [17] for details).

The Maple command *IsometryAlgebraData* sequentially computes the linear system $(E_0,\ \ldots,\ E_k)$ and at each step checks the rank of the system. When the rank stabilizes, a basis for the vector space of Killing data is determined by the solutions of the above equations. To be more precise, *IsometryAlgebraData* computes the rank, not from the covariant derivatives of the curvature but rather using symmetrized covariant derivatives of the curvature as calculated from the *YoungCurvatureTensor* command, a significantly more efficient approach.

A Lie bracket on the space of Killing data $\mathcal{K}$ can be defined by

$$[Z_1, Z_2] = \Big(X_1^b \nabla_b X_2^a - X_2^b \nabla_b X_1^a, \nabla_a X_1^c \nabla_c X_2^b - \nabla_a X_2^c \nabla_c X_1^b + X_1^c X_2^d R^b{}_{acd}(p) \Big),$$

(see Ashtekar [18]). The Lie algebra determined in this way is isomorphic to the isometry algebra Γ of g. The subalgebra defined by Killing data of the form $(0, \nabla_a X^b)$ is isomorphic to the isotropy subalgebra.

Now consider the case where $M = G/H$ and g is an invariant metric, as described in Section 2.5. By construction, we are assured that the isometry algebra of g will contain the Lie algebra of G but it may happen that the isometry algebra of g can have dimension greater than that of G. For example, it sometimes happens that the general invariant metric has constant curvature in which case the isometry algebra will be of maximal dimension, namely dimension 10. However, for any G-invariant metric on G/H, there are formulas for the curvature tensor. The curvature tensor of G-invariant metrics on G/H can be computed from the structure constants of $\mathfrak{g}$ and a given $\mathrm{ad}(\mathfrak{h})$-invariant inner product. Consequently, one can check directly whether or not the $\mathrm{ad}(\mathfrak{h})$-invariant inner product is giving a flat metric or a metric of constant curvature. See Coquereaux [19], page 74, for further details. More generally, the derivatives of the curvature tensor of the invariant metric can be computed in terms of the structure constants and therefore the dimension of the isometry algebra can be computed using the program *IsometryAlgebraData*. Combining these results, we deduce that the actual isometry algebra of a G-invariant metric can be computed from the structure constants of the Lie algebra of G, that is, there is no need to write the metric in terms of local coordinates on G/H. For examples regarding this discussion see Chapter 3, Equation (3.2), and see Chapter 7 and the appendices for explicit use of the Maple command *IsometryAlgebraData* in this context.

2.7 The Schmidt Method

The Schmidt method is a process by which we construct a Lie algebra $\mathfrak{g}$ with subalgebra $\mathfrak{h}$ where $\mathfrak{g}$ admits a realization as a Lie algebra of Killing vectors Γ on a pseudo-Riemannian manifold M with $\mathfrak{h}$ realized as the isotropy subalgebra $\Gamma_p \subset \Gamma$, $p \in M$. Note that $(\mathfrak{g}, \mathfrak{h})$ will form a Lorentzian Lie algebra-subalgebra pair. We should comment, however, that Γ may only be a subalgebra of the full isometry algebra. That is, the Schmidt method does not guarantee the constructed pair $(\mathfrak{g}, \mathfrak{h})$ is identically the full isometry-isotropy algebra-subalgebra pair for the metric on M but perhaps only contained therein (in an abstract sense). We will now summarize at the algebraic level the pertinent parts of the process considered in Schmidt [6].

Let $\mathfrak{h}$ be a subalgebra of a Lie algebra $\mathfrak{g}$. Suppose $\mathfrak{h}$ abstractly defines a subalgebra of $\mathfrak{so}\,(p,1)$ and $\mathfrak{m}$ is a reductive complement with $\dim(\mathfrak{m}) = m$. For the non-reductive case, see Fels [5] for an alternative approach. Let A_i be the adjoint representations of $h_i \in \mathfrak{h}$ restricted to $\mathfrak{m}$. For an ad($\mathfrak{h}$)-invariant inner product η on $\mathfrak{m}$, the A_i satisfy $A^T\eta + \eta A = 0$. Let A_i have entries $A_{i\alpha}{}^{\beta}$ and matrix commutators $[A_i, A_j] = C_{ij}{}^{k}A_k$ (see table 2.1). Let $h_i \in \mathfrak{h}$ and $m_\alpha \in \mathfrak{m}$ be basis elements for $\mathfrak{g}$ with brackets

$$\begin{aligned}
[h_i, h_j] &= C_{ij}{}^{k}h_k \\
[m_\alpha, h_i] &= A_{i\alpha}{}^{\beta}m_\beta \\
[m_\alpha, m_\beta] &= a_{\alpha\beta}{}^{k}h_k + b_{\alpha\beta}{}^{\gamma}m_\gamma
\end{aligned}$$

with $a_{\alpha\beta}^{\gamma}$ and $b_{\alpha\beta}^{k}$ undetermined up to the Jacobi identities being satisfied. The Jacobi identities give the following equations:

$$\begin{aligned}
&C_{[ij}{}^{\beta}C_{k]\beta}{}^{\alpha} = 0 \\
&A_{j\alpha}{}^{\beta}A_{i\beta}{}^{\gamma} - A_{i\alpha}{}^{\beta}A_{j\beta}{}^{\gamma} - C_{ij}{}^{k}A_{k\alpha}{}^{\gamma} = 0 \\
&2A_{i[\alpha}{}^{\rho}b_{\beta]\rho}^{\gamma} + 2A_{i[\alpha}{}^{\rho}a_{\beta]\rho}^{k} + b_{\alpha\beta}^{\rho}A_{i\rho}{}^{\gamma} + a_{\alpha\beta}^{r}C_{ir}{}^{k} = 0 \\
&b_{[\beta\gamma}^{\rho}b_{\alpha]\rho}{}^{\kappa} + a_{[\beta\gamma}^{k}A_{\alpha]k}{}^{\kappa} = 0
\end{aligned}$$

with square brackets indicating anti-symmetrization. The general procedure we then conform to is as follows:

i) Fix a subalgebra of the Lorentz algebra and identify the infinitesimal generators in the standard representation (see table 3.1), yielding a linear transformation from $\mathbb{R}^m$ to itself, with m the dimension of the reductive complement $\mathfrak{m}$.

ii) These transformations are taken to define the adjoint of each basis element $h_i \in \mathfrak{h}$ in $\mathfrak{g}$. We also order the basis elements of $\mathfrak{g}$ according to the sum $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$.

iii) Impose the Jacobi identities to eliminate arbitrary parameters. There will in general remain parameters in the structure equations.

iv) Find changes of basis on $\mathfrak{m}$ which leave the adjoint action of the isotropy $\mathfrak{h}$ undisturbed yet eliminate remaining inessential parameters from the structure equations.

v) Classify the Lie algebras against a standard reference such as Snobl [11].

vi) Continue this process until all inessential parameters are removed from the structure equations and all inequivalent Lie algebra pairs are discovered.

For step iv) above, we are describing transformations from the set Φ of non-degenerate linear transformations $\phi : \mathfrak{g} \to \mathfrak{g}$ such that $[\phi(m_\alpha), h_i] = [m_\alpha, h_i]$ and $\phi(h_i) = h_i$. Note Φ is given by intersecting the matrix centralizers for the adjoint of each $h_i \in \mathfrak{h}$. Therefore, the essence of the Schmidt method is to impose the Jacobi identities on $a_{\alpha\beta}^{\,k}$ and $b_{\alpha\beta}^{\,\gamma}$, resulting in a set of quadratic equations to be solved, and selecting a representation for Φ-orbits.

A Lie algebra-subalgebra pair $(\mathfrak{g}, \mathfrak{h})$ is *Lorentzian* if $\mathfrak{h}$ acts as a subalgebra of the Lorentz algebra $\mathfrak{so}(3,1)$ on $\mathfrak{g}/\mathfrak{h}$ and $\dim(\mathfrak{g}/\mathfrak{h}) \leq 4$. Then the above procedure permits the classification of all reductive Lorentzian Lie algebra-subalgebra pairs $(\mathfrak{g}, \mathfrak{h})$. We classify the cases in which $\mathfrak{g}$ has dimension 6 with $\mathfrak{h}$ dimension two or three as well as for $\mathfrak{g}$ having dimension 7 with $\mathfrak{h}$ dimension three in Chapter 3.

2.7.1 Applying the Schmidt Method

By Remark 2.3.12, if a local slice exists at a point, then an isotropy preserving local slice S exists through that point. Then Corollary 2.3.14 shows that the isotropy subgroup at $p \in M$ conjugates to isotropy subgroups in a neighborhood U of p. In U, isometry-isotropy algebra-subalgebra pairs are equivalent. Since the manifold is diffeomorphic to the Cartesian product $(S \cap U) \times G/G_p$, the Schmidt method applies to the homogeneous space G/G_p. Theorem 2.3.13 shows G/G_p is diffeomorphic to the orbit $O_G(p)$, and thus the isometry-isotropy algebra-subalgebra pair for G/G_p is equivalent to that on M. Then to classify simple G spacetimes (M, g) with symmetry, we may classify Lie algebra-subalgebra pairs associated to isometry groups G and isotropy subgroups $H = G_p \subset G$ by applying the Schmidt method, working at the algebraic level. Conversely, to ascertain whether or not a given isometry-isotropy pair is calculable by way of the Schmidt method, conditions under which a local slice exists must be determined. See example 2.4.6 for a method of determining when such a pair does not admit a local slice and therefore not simple G.

2.7.2 Subalgebras of $\mathfrak{so}(3,1)$

Lorentzian Lie algebra-subalgebra pairs by definition have subalgebra $\mathfrak{h}$ abstractly defining a subalgebra of $\mathfrak{so}(3,1)$. In Winternitz [20], the subalgebras of $\mathfrak{so}(3,1)$ up to conjugation have been classified and are given in Table 2.1.

TABLE 2.1: Subalgebras of $\mathfrak{so}\,(3,1)$ considered as 4×4 matrices which preserve the Minkowski spacetime, labeled $F1-F15$, as classified in Winternitz [20].

$F1$:	$\{B_1, B_2, B_3, B_4, B_5, B_6\}$	$F9$:	$\{B_1, B_2\}$
$F2$:	$\{B_1, B_2, B_3, B_4\}$	$F10$:	$\{B_3, B_4\}$
$F3$:	$\{R_x, R_y, R_z\}$	$F11$:	$\{B\,(\theta)\}$
$F4$:	$\{R_z, K_x, K_y\}$	$F12$:	$\{R_z\}$
$F5$:	$\{B\,(\theta), B_3, B_4\}$	$F13$:	$\{K_z\}$
$F6$:	$\{B_1, B_3, B_4\}$	$F14$:	$\{R_y + K_z\}$
$F7$:	$\{B_2, B_3, B_4\}$	$F15$:	$\{0\}$
$F8$:	$\{B_2, B_3\}$		

$$R_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad R_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad R_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad K_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$B_1 = 2R_z \qquad B_2 = -2K_z \qquad B_3 = -R_y - K_z$$
$$B_4 = R_x - K_y \qquad B_5 = R_y - K_x \qquad B_6 = R_x + K_y \qquad B\,(\theta) = \cos(\theta)\,R_z - \sin(\theta)\,K_z$$

CHAPTER 3

THE CLASSIFICATION OF LORENTZIAN PAIRS

A Lie algebra-subalgebra pair $(\mathfrak{g}, \mathfrak{h})$ is *Lorentzian* if under the adjoint action $\mathfrak{h}$ acts as a subalgebra of the Lorentz algebra $\mathfrak{so}(k,1)$ on $\mathfrak{g}/\mathfrak{h}$ and $\dim(\mathfrak{g}/\mathfrak{h}) \leq k+1$, for $k = 2, 3$. In this chapter we shall find all possible non-equivalent Lorentzian Lie algebra-subalgebra pairs $(\mathfrak{g}, \mathfrak{h})$ (see definition 2.1.2 regarding the equivalence of pairs). We are specifically interested in *maximal* Lorentzian pairs. A Lorentzian pair $(\mathfrak{g}, \mathfrak{h})$ is maximal if the isometry algebra $\tilde{\mathfrak{g}}$ of the general $\mathrm{ad}(\mathfrak{h})$-invariant Lorentzian inner product g on $\mathfrak{g}/\mathfrak{h}$ is precisely $\mathfrak{g}$. For example, if the curvature tensor of g is constant, then $\dim(\tilde{\mathfrak{g}}) = 10$ and the pair $(\mathfrak{g}, \mathfrak{h})$ is then disregarded (see the end of Section 2.5). The full listing of Lorentzian pairs will be presented in Appendix A.1. All labeling of Lorentzian pairs in this dissertation take the form $[a, b, c]$, where $a = \dim(\mathfrak{g})$, $b = \dim(\mathfrak{g}/\mathfrak{h})$, and for fixed a and b, c simply labels the entries, beginning with 1.

The possibilities for the dimensions of Lorentzian pairs for Lie algebras of dimension three through seven with non-trivial isotropy are

$$\dim(\mathfrak{g}, \mathfrak{h}) = (3,1),\ (4,1),\ (5,1),\ (5,2),\ (6,2),\ (6,3),\ \text{and } (7,3).$$

Table 3.1 gives the list of adjoint representations of the isotropy subalgebras of the Lorentzian pairs for each of these cases. The matrix representations define subalgebras of $\mathfrak{so}(3,1)$, classified as $F1$ through $F14$. Note that $F15$ refers to the trivial subalgebra. Note that the classification of Lorentzian pairs for the case of trivial isotropy is equivalent to the classification of three- and four-dimensional Lie algebras, which has been given in Šnobl [11]. These are labeled $[3,3,1]$ through $[3,3,9]$ and $[4,4,1]$ through $[4,4,23]$ in Appendix A.1.

For the cases where $(\mathfrak{g}, \mathfrak{h})$ is reductive, $(3,1)$ and $(4,1)$ were treated in Bowers [4] and are labeled in the appendix as $[3,2,1]$ through $[3,2,5]$ for $(3,1)$ and $[4,3,1]$ through $[4,3,20]$ for $(4,1)$. Cases $(5,1)$ and $(5,2)$ were treated in Rozum [10] and are labeled $[5,4,1]$ through $[5,4,11]$ in the appendix. The $\mathrm{ad}(\mathfrak{h})$-invariant inner product in the $(5,2)$ case admits

TABLE 3.1: Standard forms of adjoint representations of isotropy subalgebras and the isotropy type of each, thought of as abstractly defining subalgebras of $\mathfrak{so}(3,1)$.

$(\dim \mathfrak{g}, \dim \mathfrak{h})$	**Isotropy**	**Isotropy Type**
$(3,1)$	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$F12$, $F13$, resp.
$(4,1)$	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$	$F12$, $F13$, $F14$, resp.
$(5,1)$	$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$F12$, $F13$, $F14$, resp.
$(6,3)$	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$	$F3$
$(6,3)$	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$F4$
$(6,2)$	$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$	$F9$
$(6,2)$	$\begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$	$F10$
$(7,3)$	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	F3
$(7,3)$	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	F4
$(7,3)$	$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$	$F6$

additional symmetries and is therefore disregarded as a non-maximal Lorentzian pair. All non-reductive Lorentzian pairs were treated in Fels [5] and are labeled $[5,4,-1]$ through $[5,4,-6]$, in addition to $[6,4,-1]$, in the appendix. The minus sign indicates these are non-reductive cases. The non-reductive Lorentzian pair $A5^*$ of [5] (note $\mathfrak{g}$ is seven-dimensional) admits an invariant metric of constant curvature having a ten-dimensional isometry algebra and is thus excluded from the database. Fels used a method due to Élie Cartan for the non-reductive cases and the reader is referred to the article for further information and their approach. Bowers and Rozum both used the Schmidt method in their analysis.

The remainder of the chapter will employ the Schmidt method to discover all reductive Lorentzian pairs for the cases $(6,2)$, $(6,3)$, and $(7,3)$. The full results can be found in Appendix A.1.

3.1 Six-dimensional Lie algebras on three-dimensional quotients

We investigate pairs $(\mathfrak{g},\mathfrak{h})$ involving a six-dimensional Lie algebra $\mathfrak{g}$ with a three-dimensional subalgebra $\mathfrak{h}$ admitting a reductive complement m, where $\mathfrak{g}/\mathfrak{h}$ admits a three-dimensional $\mathrm{ad}(\mathfrak{h})$-invariant inner product having Riemannian or Lorentzian signature. The only possibilities are $\mathfrak{h}$ acting as $\mathfrak{so}(3)$ or as $\mathfrak{so}(2,1)$. These are subalgebras $F3$ and $F4$ respectively of $\mathfrak{so}(3,1)$. The storyline for each case is very similar in word and spirit. The results of this section give the database entries $[6,3,1]$, $[6,3,2]$, $[6,3,3]$, $[6,3,4]$, $[6,3,5]$, and $[6,3,6]$.

3.1.1 $F3$

The basis in this case is defined by the following matrices

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

designated as e_4, e_5, and e_6 of $\mathfrak{g}$, respectively. Observe that $[e_4,e_5]=e_6$, $[e_4,e_6]=-e_5$, and $[e_5,e_6]=e_4$. Then we have the following structure equations by assuming the above matrices define the adjoint action of e_4, e_5, and e_6 respectively, restricted to a reductive complement $m=\mathrm{span}(\{e_1,e_2,e_3\})$:

$$\begin{aligned}
&[e_1,e_4]=-e_2, && [e_2,e_4]=e_1, && [e_3,e_4]=0, && [e_4,e_5]=e_6,\\
&[e_1,e_5]=-e_3, && [e_2,e_5]=0, && [e_3,e_5]=e_1, && [e_4,e_6]=-e_5,\\
&[e_1,e_6]=0, && [e_2,e_6]=-e_3, && [e_3,e_6]=e_2, && [e_5,e_6]=e_4.
\end{aligned}$$

The remaining structure equations to be determined are:

$$\begin{aligned}
[e_1, e_2] &= a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6, \\
[e_1, e_3] &= b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5 + b_6e_6, \\
[e_2, e_3] &= c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5 + c_6e_6.
\end{aligned}$$

However, imposing the Jacobi identities reduces the number of unknowns and we arrive at the following:

$$[e_1, e_2] = -b_2e_3 + c_6e_4, \ [e_1, e_3] = b_2e_2 + c_6e_5, \ [e_2, e_3] = -b_2e_1 + c_6e_6, \tag{3.1}$$

noting all Jacobi identities are thus satisfied. We make the following change of basis:

$$\begin{aligned}
&\tilde{e}_1 = e_1 + \tfrac{1}{2}b_2e_6, \quad &&\tilde{e}_2 = e_2 - \tfrac{1}{2}b_2e_5, \quad &&\tilde{e}_3 = e_3 + \tfrac{1}{2}b_2e_4, \\
&\tilde{e}_4 = e_4, &&\tilde{e}_5 = e_5, &&\tilde{e}_6 = e_6.
\end{aligned}$$

If we compute the structure equations for this new basis and immediately drop the tilde, we arrive at the following multiplication table for the Lie algebra, where $a = c_6 + b_2^2/4$:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	ae_4	ae_5	$-e_2$	$-e_3$	.
e_2		.	ae_6	e_1	.	$-e_3$
e_3			.	.	e_1	e_2
e_4				.	e_6	$-e_5$
e_5					.	e_4
e_6						.

Observe the isotropy and its action on the reductive complement remain undisturbed by this change of basis.

We investigate three cases: $a > 0$, $a = 0$, and $a < 0$.

Case 1: $a > 0$. Then the change of basis

$$\{\frac{e_1}{\sqrt{a}}, \frac{e_2}{\sqrt{a}}, \frac{e_3}{\sqrt{a}}, e_4, e_5, e_6\}$$

gives the following structure equations

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_4	e_5	$-e_2$	$-e_3$	.
e_2		.	e_6	e_1	.	$-e_3$
e_3			.	.	e_1	e_2
e_4				.	e_6	$-e_5$
e_5					.	e_4
e_6						.

If we change basis by

$$\left\{\frac{1}{2}(e_1 + e_6), \frac{1}{2}(e_2 - e_5), \frac{1}{2}(e_3 + e_4), -\frac{1}{2}(e_1 - e_6), \frac{1}{2}(e_2 + e_5), \frac{1}{2}(e_3 - e_4)\right\}$$

we see this Lie algebra is $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_3	$-e_2$	.	.	.
e_2		.	e_1	.	.	.
e_3			.	.	.	.
e_4				.	e_6	$-e_5$
e_5					.	e_4
e_6						.

The isotropy is $\{e_3 - e_6, -e_2 + e_5, e_1 + e_4\}$. These structure equations are stored in the database as entry $[6, 3, 1]$.

Case 2: $a = 0$. This gives the structure equations as follows:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	.	$-e_2$	$-e_3$	.
e_2		.	.	e_1	.	$-e_3$
e_3			.	.	e_1	e_2
e_4				.	e_6	$-e_5$
e_5					.	e_4
e_6						.

These structure equations are stored as $[6,3,2]$ in the database. As seen, the Lie algebra admits a proper Levi decomposition with abelian radical given by $\{e_1, e_2, e_3\}$ and semisimple part $\{e_4, e_5, e_6\}$. Thus the Lie algebra is the Euclidean Lie algebra $\mathfrak{euc}(3)$. This is labeled as $\mathfrak{so}(3,\mathbb{R}) \ni 3\mathfrak{n}_{1,1}$ in Šnobl [11].

<u>Case 3:</u> $a < 0$. Then the change of basis

$$\{\frac{e_1}{\sqrt{-a}}, \frac{e_2}{\sqrt{-a}}, \frac{e_3}{\sqrt{-a}}, e_4, e_5, e_6\}$$

gives the following structure equations

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	$-e_4$	$-e_5$	$-e_2$	$-e_3$	.
e_2		.	$-e_6$	e_1	.	$-e_3$
e_3			.	.	e_1	e_2
e_4				.	e_6	$-e_5$
e_5					.	e_4
e_6						.

If we change basis by

$$\{-e_5, -e_6, e_4, -e_3, -e_1, -e_2\}$$

the Lie algebra is in a standard form of $\mathfrak{so}(3,1)$:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_3	$-e_2$	e_5	$-e_4$	.
e_2		.	e_1	e_6	.	$-e_4$
e_3			.	.	e_6	$-e_5$
e_4				.	$-e_1$	$-e_2$
e_5					.	$-e_3$
e_6						.

The isotropy has become $\{e_3, -e_1, -e_2\}$. These structure equations are labeled $[6, 3, 3]$.

The results of this subsection give Lorentzian pairs $[6, 3, 1]$, $[6, 3, 2]$, $[6, 3, 3]$.

3.1.2 $F4$

The basis in this case is defined by the following matrices

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

designated as e_4, e_5, and e_6 of $\mathfrak{g}$, respectively. Observe that $[e_4, e_5] = e_6$, $[e_4, e_6] = -e_5$, and $[e_5, e_6] = -e_4$. Then we have the following structure equations by assuming the above matrices define the adjoint action of e_4, e_5, and e_6 respectively, restricted to a reductive complement $m = \mathrm{span}(\{e_1, e_2, e_3\})$:

$$\begin{array}{llll} [e_1, e_4] = -e_2, & [e_2, e_4] = e_1, & [e_3, e_4] = 0, & [e_4, e_5] = e_6, \\ [e_1, e_5] = -e_3, & [e_2, e_5] = 0, & [e_3, e_5] = -e_1, & [e_4, e_6] = -e_5, \\ [e_1, e_6] = 0, & [e_2, e_6] = -e_3, & [e_3, e_6] = -e_2, & [e_5, e_6] = -e_4, \end{array}$$

The remaining structure equations to be determined are:

$$[e_1, e_2] = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6,$$

$$[e_1, e_3] = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5 + b_6e_6,$$

$$[e_2, e_3] = c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5 + c_6e_6.$$

Imposing the Jacobi identities reduces the number of unknowns and we arrive at the following:

$$[e_1, e_2] = b_2e_3 + a_4e_4,\ [e_1, e_3] = b_2e_2 + a_4e_5,\ [e_2, e_3] = -b_2e_1 + a_4e_6,$$

noting all Jacobi identities are thus satisfied. Make the change of basis,

$$\begin{array}{lll} \tilde{e}_1 = e_1 - \frac{1}{2}b_2e_6, & \tilde{e}_2 = e_2 + \frac{1}{2}b_2e_5, & \tilde{e}_3 = e_3 + \frac{1}{2}b_2e_4, \\ \tilde{e}_4 = e_4, & \tilde{e}_5 = e_5, & \tilde{e}_6 = e_6. \end{array}$$

Dropping the tilde and computing the structure equations for this new basis, setting $a = a_4 - b_2^2/4$, we arrive at the following multiplication table for the Lie algebra:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	ae_4	ae_5	$-e_2$	$-e_3$	.
e_2		.	ae_6	e_1	.	$-e_3$
e_3			.	.	$-e_1$	$-e_2$
e_4				.	e_6	$-e_5$
e_5					.	$-e_4$
e_6						.

We investigate three cases: $a < 0$, $a = 0$, and $a > 0$.

Case 1: $a < 0$. Then the change of basis

$$\{\frac{e_1}{\sqrt{-a}}, \frac{e_2}{\sqrt{-a}}, \frac{e_3}{\sqrt{-a}}, e_4, e_5, e_6\}$$

gives the following structure equations

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	$-e_4$	$-e_5$	$-e_2$	$-e_3$	.
e_2		.	$-e_6$	e_1	.	$-e_3$
e_3			.	.	$-e_1$	$-e_2$
e_4				.	e_6	$-e_5$
e_5					.	$-e_4$
e_6						.

If we change basis by

$$\Big\{\frac{1}{2}(e_1 - e_6), \frac{1}{2}(e_2 - e_3 - e_4 + e_5), \frac{-1}{4}(e_2 + e_3 + e_4 + e_5), \frac{1}{2}(e_1 + e_6),$$
$$\frac{1}{2}(e_2 + e_3 - e_4 - e_5), \frac{-1}{4}(e_2 - e_3 + e_4 - e_5)\Big\}$$

we see this Lie algebra is $\mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)$:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_2	$-e_3$	.	.	.
e_2		.	$-e_1$	.	.	.
e_3			.	.	.	.
e_4				.	e_5	$-e_6$
e_5					.	$-e_4$
e_6						.

These structure equations are stored in the database as entry $[6, 3, 4]$, with isotropy given by $\{\frac{1}{2}e_2 - e_3 - \frac{1}{2}e_5 - e_6, \frac{1}{2}e_2 - e_3 - \frac{1}{2}e_5 + e_6, -e_1 + e_4\}$.

<u>Case 2:</u> $a = 0$. The structure equations are then as follows:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	.	$-e_2$	$-e_3$	.
e_2		.	.	e_1	.	$-e_3$
e_3			.	.	$-e_1$	$-e_2$
e_4				.	e_6	$-e_5$
e_5					.	$-e_4$
e_6						.

These structure equations are stored as $[6, 3, 5]$ in the database. Note from the table above that the Lie algebra admits a proper Levi decomposition with abelian radical given by $\{e_1, e_2, e_3\}$ and semisimple part $\{e_4, e_5, e_6\}$. Thus the Lie algebra is the Euclidean Lie algebra $\mathfrak{euc}(2, 1)$. This is notated $\mathfrak{sl}(2, \mathbb{R}) \ni 3\mathfrak{n}_{1,1}$ in Šnobl [11], recalling that $\mathfrak{sl}(2, \mathbb{R})$ and $\mathfrak{so}(2, 1)$ are isomorphic.

Case 3: $a > 0$. Then the change of basis

$$\{\frac{e_1}{\sqrt{a}}, \frac{e_2}{\sqrt{a}}, \frac{e_3}{\sqrt{a}}, e_4, e_5, e_6\}$$

gives the following structure equations

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_4	e_5	$-e_2$	$-e_3$	.
e_2		.	e_6	e_1	.	$-e_3$
e_3			.	.	$-e_1$	$-e_2$
e_4				.	e_6	$-e_5$
e_5					.	$-e_4$
e_6						.

If we change basis by

$$\{e_2, e_1, -e_4, e_3, e_6, e_5\}$$

this Lie algebra is seen to be $\mathfrak{so}(3, 1)$ with isotropy $\{-e_3, e_6, e_5\}$; these structure equations are stored in the database as entry $[6, 3, 6]$.

The results of this subsection give the database entries $[6, 3, 4]$, $[6, 3, 5]$, $[6, 3, 6]$.

3.2 Six-dimensional Lie algebras on four-dimensional quotients

We now turn our attention to pairs $(\mathfrak{g}, \mathfrak{h})$ involving a six-dimensional Lie algebra $\mathfrak{g}$ with a two-dimensional subalgebra $\mathfrak{h}$ admitting a reductive complement m, where $\mathfrak{g}/\mathfrak{h}$ admits an $\mathrm{ad}(\mathfrak{h})$-invariant inner product having Lorentzian signature. Note that $\mathfrak{h}$ acts as a two-dimensional subalgebra of $\mathfrak{so}(3,1)$. The two-dimensional subalgebras of $\mathfrak{so}(3,1)$ are $F8$, $F9$, and $F10$, as classified in Winternitz [20]. The results of this section give the database entries $[6, 4, 1]$, $[6, 4, 2]$, $[6, 4, 3]$, $[6, 4, 4]$, $[6, 4, 5]$, and $[6, 4, 6]$.

3.2.1 $F8$

The basis in this case is defined by the following matrices

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

designated as e_5 and e_6 of $\mathfrak{g}$, respectively. Observe that $[e_5, e_6] = e_6$. By assuming the above matrices define the adjoint action of e_5 and e_6 respectively, restricted to a reductive complement $m = \mathrm{span}(\{e_1, e_2, e_3, e_4\})$, we obtain the following structure equations:

$$\begin{aligned} &[e_1, e_5] = 0, && [e_1, e_6] = -e_3 + e_4, && [e_2, e_5] = 0, && [e_2, e_6] = 0, \\ &[e_3, e_5] = e_4, && [e_3, e_6] = e_1, && [e_4, e_5] = e_3, && [e_4, e_6] = e_1, \\ &[e_5, e_6] = e_6. \end{aligned}$$

The remaining structure equations to be determined are:

$$[e_1, e_2] = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6,$$

$$[e_1, e_3] = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + b_5 e_5 + b_6 e_6,$$

$$[e_1, e_4] = d_1 e_1 + d_2 e_2 + d_3 e_3 + d_4 e_4 + d_5 e_5 + d_6 e_6,$$

$$[e_2, e_3] = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5 + c_6 e_6,$$

$$[e_2, e_4] = g_1 e_1 + g_2 e_2 + g_3 e_3 + g_4 e_4 + g_5 e_5 + g_6 e_6,$$
$$[e_3, e_4] = h_1 e_1 + h_2 e_2 + h_3 e_3 + h_4 e_4 + h_5 e_5 + h_6 e_6.$$

Imposing the Jacobi identities reduces the number of unknowns and we arrive at the following

$$[e_1, e_2] = a_1 e_1,\ [e_1, e_3] = -h_1 e_4,\ [e_2, e_3] = -a_1 e_3,$$
$$[e_1, e_4] = -h_1 e_3,\ [e_2, e_4] = -a_1 e_4,\ [e_3, e_4] = h_1 e_1,$$

noting all Jacobi identities are satisfied only if $a_1 = 0$ or $h_1 = 0$. However, the invariant metric admits additional symmetries. Indeed the $\mathrm{ad}(\mathfrak{h})$-invariant inner product on $\mathfrak{g}/\mathfrak{h}$ can be seen to be given by

$$g = A\sigma^1 \otimes \sigma^1 + B\sigma^2 \otimes \sigma^2 + A\sigma^3 \otimes \sigma^3 - A\sigma^4 \otimes \sigma^4 \tag{3.2}$$

where A, B are constants and the 1-forms $\{\sigma^i\}_{i=1\ldots4}$ form the dual basis to the basis $\{e_i + \mathfrak{h}\}$ of $\mathfrak{g}/\mathfrak{h}$. If $a_1 = h_1 = 0$, the metric is flat. For $a_1 \neq 0, h_1 = 0$, the metric has a ten-dimensional isometry algebra and constant sectional curvature $K = -1/B$. And for $a_1 = 0, h_1 \neq 0$, the metric has a seven-dimensional isometry algebra with classification $[7, 4, 4]$, found in Section 3.3.

Therefore there are no six-dimensional Lorentzian pairs with two-dimensional isotropy of type $F8$ whose invariant metric does not admit additional symmetries.

3.2.2 $F9$

The basis in this case is defined by the following matrices

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

designated as e_5 and e_6 of $\mathfrak{g}$, respectively. Observe that $[e_5, e_6] = 0$. By assuming the above matrices define the adjoint action of e_5 and e_6 respectively, restricted to a reductive complement $m = \text{span}(\{e_1, e_2, e_3, e_4\})$, we obtain the following structure equations:

$$\begin{aligned}
&[e_1, e_5] = -e_2, \quad &&[e_1, e_6] = 0, \quad &&[e_2, e_5] = e_1, \quad &&[e_2, e_6] = 0,\\
&[e_3, e_5] = 0, \quad &&[e_3, e_6] = e_4, \quad &&[e_4, e_5] = 0, \quad &&[e_4, e_6] = e_3,\\
&[e_5, e_6] = 0.
\end{aligned}$$

The remaining structure equations to be determined are:

$$\begin{aligned}
[e_1, e_2] &= a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6,\\
[e_1, e_3] &= b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5 + b_6e_6,\\
[e_1, e_4] &= d_1e_1 + d_2e_2 + d_3e_3 + d_4e_4 + d_5e_5 + d_6e_6,\\
[e_2, e_3] &= c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5 + c_6e_6,\\
[e_2, e_4] &= g_1e_1 + g_2e_2 + g_3e_3 + g_4e_4 + g_5e_5 + g_6e_6,\\
[e_3, e_4] &= h_1e_1 + h_2e_2 + h_3e_3 + h_4e_4 + h_5e_5 + h_6e_6.
\end{aligned}$$

Imposing the Jacobi identities reduces the number of unknowns and we arrive at the following:

$$\begin{aligned}
&[e_1, e_2] = a_5e_5,\ [e_1, e_3] = 0,\ [e_2, e_3] = 0,\\
&[e_1, e_4] = 0,\ [e_2, e_4] = 0,\ [e_3, e_4] = h_6e_6,
\end{aligned}$$

noting all Jacobi identities are satisfied. Setting $a = a_5$ and $b = h_6$ we arrive at the following structure equations:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	ae_5	.	.	$-e_2$	.
e_2		.	.	.	e_1	.
e_3			.	be_6	.	e_4
e_4				.	.	e_3
e_5					.	.
e_6						.

The case $a = b = 0$ admits an invariant metric with additional symmetries. Specifically, the $\mathrm{ad}(\mathfrak{h})$-invariant inner product on $\mathfrak{g}/\mathfrak{h}$ for the case $a = b = 0$ can be seen to be given by

$$g = A\sigma^1 \otimes \sigma^1 + A\sigma^2 \otimes \sigma^2 + B\sigma^3 \otimes \sigma^3 - B\sigma^4 \otimes \sigma^4$$

where A, B are constants, and has vanishing curvature tensor.

We will thus assume $a^2 + b^2 \neq 0$. Using the change of basis $\{ue_1, ue_2, ve_3, ve_4, e_5, e_6\}$ for nonzero u and v, we get the following table:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	au^2e_5	.	.	$-e_2$	.
e_2		.	.	.	e_1	.
e_3			.	bv^2e_6	.	e_4
e_4				.	.	e_3
e_5					.	.
e_6						.

We break into cases based on whether or not a and b are positive, negative, or zero, assuming $a^2 + b^2 \neq 0$. As we shall see, all cases give decomposable Lie algebras.

<u>Case 1:</u> $a > 0$ with $b > 0$ or $b < 0$. Let $u = 1/\sqrt{a}$ and $v = 1/\sqrt{|b|}$. Then the change of basis

$$\left\{\frac{e_1}{\sqrt{a}}, \frac{e_2}{\sqrt{a}}, \frac{e_3}{\sqrt{|b|}}, \frac{e_4}{\sqrt{|b|}}, e_5, e_6\right\}$$

gives the following structure equations, where $\varepsilon = b/|b| = \pm 1$:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_5	.	.	$-e_2$	.
e_2		.	.	.	e_1	.
e_3			.	εe_6	.	e_4
e_4				.	.	e_3
e_5					.	.
e_6						.

For either choice of ε the Lie algebras are isomorphic: making the change of basis

$$\big\{-e_5, e_2, e_1, e_3+e_4-e_6, -e_3-e_4, -e_4+e_6\big\}$$

and

$$\big\{-e_5, e_1, -e_2, e_3-e_4-e_6, e_3-e_6, e_3-e_4\big\}$$

respectively for $\varepsilon = 1$ and $\varepsilon = -1$, we see this Lie algebra is $\mathfrak{so}(3) \oplus \mathfrak{so}(2,1)$:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_3	$-e_2$	.	.	.
e_2		.	e_1	.	.	.
e_3			.	.	.	.
e_4				.	e_5	$-e_6$
e_5					.	$-e_4$
e_6						.

These structure equations are stored in the database as entry $[6,4,1]$. The isotropy is given by $\{-e_1, -e_4-e_5\}$.

<u>Case 2:</u> $a < 0$ with $b > 0$ or $b < 0$. Let $u = 1/\sqrt{-a}$ and $v = 1/\sqrt{|b|}$. Then the change of basis

$$\left\{\frac{e_1}{\sqrt{-a}}, \frac{e_2}{\sqrt{-a}}, \frac{e_3}{\sqrt{|b|}}, \frac{e_4}{\sqrt{|b|}}, e_5, e_6\right\}$$

gives the following structure equations, where $\varepsilon = b/|b| = \pm 1$:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	$-e_5$	.	.	$-e_2$	.
e_2		.	.	.	e_1	.
e_3			.	εe_6	.	e_4
e_4				.	.	e_3
e_5					.	.
e_6						.

For either choice of ε the Lie algebras are isomorphic. If we change basis by

$$\{e_3 - e_4 - e_6, -e_4 - e_6, e_3 - e_4, e_1 - e_2 - e_5, e_1 - e_5, -e_2 - e_5\}$$

and

$$\{e_1 - e_2 - e_5, -e_1 + e_5, e_2 + e_5, -e_3 + e_4 + e_6, e_3 - e_4, e_3 - e_6\}$$

for $\varepsilon = 1$ and $\varepsilon = -1$ respectively, we see this Lie algebra is $\mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)$:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_2	$-e_3$	.	.	.
e_2		.	$-e_1$	.	.	.
e_3			.	.	.	.
e_4				.	e_5	$-e_6$
e_5					.	$-e_4$
e_6						.

and these structure equations are stored in the database as entry $[6, 4, 2]$. The isotropy is given by $\{e_4 - e_5 - e_6, -e_1 + e_3\}$.

Case 3: $a = 0$, with $b > 0$ or $b < 0$. Let $v = 1/\sqrt{|b|}$. Then we have the following structure equations, where $\varepsilon = b/|b| = \pm 1$:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	.	.	$-e_2$	.
e_2		.	.	.	e_1	.
e_3			.	εe_6	.	e_4
e_4				.	.	e_3
e_5					.	.
e_6						.

If we change basis by

$$\Big\{ -e_1, -e_2, e_1 + e_2 - e_5, -e_6, -\varepsilon\frac{1}{2}(e_3 + e_4), e_3 - e_4\Big\}$$

this Lie algebra is seen to be $\mathfrak{s}_{3,3}\big|_{\alpha=0} \oplus \mathfrak{so}(2,1)$, where $\mathfrak{s}_{3,3}\big|_{\alpha=0}$ is $\mathfrak{s}_{3,3}$ with $\alpha = 0$ from Šnobl [11]:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	e_2	.	.	.
e_2		.	$-e_1$	.	.	.
e_3			.	.	.	.
e_4				.	e_5	$-e_6$
e_5					.	$-e_4$
e_6						.

These structure equations are stored in the database as entry $[6, 4, 3]$. The isotropy is given by $\{-e_1 - e_2 - e_3, -e_4\}$.

<u>Case 4:</u> $a > 0$ with $b = 0$. Let $u = 1/\sqrt{a}$. Then the change of basis

$$\Big\{\frac{e_1}{\sqrt{a}}, \frac{e_2}{\sqrt{a}}, e_3, e_4, e_5, e_6\Big\}$$

gives the following structure equations:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_5	.	.	$-e_2$	.
e_2		.	.	.	e_1	.
e_3			.	.	.	e_4
e_4				.	.	e_3
e_5					.	.
e_6						.

If we change basis by

$$\big\{ -e_3 - e_4, -e_3 + e_4, -e_4 - e_6, e_1, e_2, e_5 \big\}$$

this Lie algebra is seen to be $\mathfrak{s}_{3,1}\big|_{a=-1} \oplus \mathfrak{so}(3)$:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	$-e_1$	.	.	.
e_2		.	e_2	.	.	.
e_3			.	.	.	.
e_4				.	e_6	$-e_5$
e_5					.	e_4
e_6						.

These structure equations are stored in the database as entry $[6, 4, 4]$ with the isotropy now $\{e_6, \frac{1}{2}e_1 - \frac{1}{2}e_2 - e_3\}$.

<u>Case 5:</u> $a < 0$ with $b = 0$. Let $u = 1/\sqrt{-a}$. Then the change of basis

$$\left\{ \frac{e_1}{\sqrt{-a}}, \frac{e_2}{\sqrt{-a}}, e_3, e_4, e_5, e_6 \right\}$$

gives the following structure equations:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	$-e_5$	.	.	$-e_2$	.
e_2		.	.	.	e_1	.
e_3			.	.	.	e_4
e_4				.	.	e_3
e_5					.	.
e_6						.

If we change basis by

$$\left\{e_3 + e_4, e_3 - e_4, -e_3 - e_6, -e_1 + e_2 + e_5, e_2 + e_5, -e_1 + e_5\right\}$$

we see this Lie algebra to be $\mathfrak{s}_{3,1}\big|_{a=-1} \oplus \mathfrak{so}(2,1)$:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	$-e_1$	.	.	.
e_2		.	e_2	.	.	.
e_3			.	.	.	.
e_4				.	e_5	$-e_6$
e_5					.	$-e_4$
e_6						.

These structure equations are stored in the database as entry $[6, 4, 5]$. The isotropy is given by $\{-e_4 + e_5 + e_6, -\frac{1}{2}e_1 - \frac{1}{2}e_2 - e_3\}$.

The results of this subsection give the database entries $[6, 4, 1]$, $[6, 4, 2]$, $[6, 4, 3]$, $[6, 4, 4]$, $[6, 4, 5]$.

3.2.3 $F10$

The basis in this case is defined by the following matrices

$$\begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

designated as e_5 and e_6 of $\mathfrak{g}$, respectively. Observe that $[e_5, e_6] = 0$. By assuming the above matrices define the adjoint action of e_5 and e_6 respectively, restricted to a reductive complement $m = \text{span}(\{e_1, e_2, e_3, e_4\})$, we obtain the following structure equations:

$$\begin{array}{llll} [e_1, e_5] = -e_3 + e_4, & [e_1, e_6] = 0, & [e_2, e_5] = 0, & [e_2, e_6] = -e_3 + e_4, \\ [e_3, e_5] = e_1, & [e_3, e_6] = e_2, & [e_4, e_5] = e_1, & [e_4, e_6] = e_2, \\ [e_5, e_6] = 0. & & & \end{array}$$

The remaining structure equations to be determined are:

$$[e_1, e_2] = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6,$$

$$[e_1, e_3] = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5 + b_6e_6,$$

$$[e_1, e_4] = d_1e_1 + d_2e_2 + d_3e_3 + d_4e_4 + d_5e_5 + d_6e_6,$$

$$[e_2, e_3] = c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5 + c_6e_6,$$

$$[e_2, e_4] = g_1e_1 + g_2e_2 + g_3e_3 + g_4e_4 + g_5e_5 + g_6e_6,$$

$$[e_3, e_4] = h_1e_1 + h_2e_2 + h_3e_3 + h_4e_4 + h_5e_5 + h_6e_6.$$

Imposing the Jacobi identities reduces the number of unknowns and we arrive at the following:

$$[e_1, e_2] = -a_4e_3 + a_4e_4,$$

$$[e_1, e_3] = -h_4e_1 + a_4e_2 - d_4e_3 + d_4e_4 + d_5e_5 - (a_4h_4 - g_5)e_6,$$

$$[e_2, e_3] = -a_4e_1 - e_2h_4 - e_3g_4 + e_4g_4 + e_5g_5 + e_6g_6,$$

$$[e_1, e_4] = -h_4e_1 + a_4e_2 - d_4e_3 + d_4e_4 + d_5e_5 - (a_4h_4 - g_5)e_6,$$
$$[e_2, e_4] = -a_4e_1 - e_2h_4 - e_3g_4 + e_4g_4 + e_5g_5 + e_6g_6,$$
$$[e_3, e_4] = -e_3h_4 + e_4h_4,$$

noting all Jacobi identities are satisfied. Changing basis to $\{e_3 - e_4, e_5, e_6, e_1 + a_4e_6, e_2, e_3 + (e_3 - e_4) - d_4e_5 - g_4e_6\}$ gives the following multiplication table, where we relabeled by $a = h_4$, $b = a_4$, $c = d_5$, $d = g_5$, $f = a_4^2 + g_6$:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	.	.	.	$-ae_1$
e_2		.	.	e_1	.	$be_3 - e_4$
e_3			.	.	e_1	$-e_5$
e_4				.	.	$ce_2 + de_3 - ae_4$
e_5					.	$de_2 + fe_3 - be_4 - ae_5$
e_6						.

These structure equations are stored in the database as $[6, 4, 6]$, with isotropy $\{e_2, e_3\}$. A vector field system giving this six-dimensional abstract Lie algebra with five parameters is found as (33.54) in Petrov [2] (see Section 5.4). The results of this subsection give only the database entry $[6, 4, 6]$ as this Lie algebra is particularly difficult to break into cases further. As such, the case splitting will be a future project.

However, observe that the nilradical is $\{e_1, e_2, e_3, e_4, e_5\}$ in this new basis and is on display in the multiplication table for $[6, 4, 6]$ just above. Furthermore, the nilradical is identical to the five-dimensional Lie algebra $\mathfrak{n}_{5,3}$ found in Šnobl [11]. Also, $[6, 4, 6]$ is indecomposable. Therefore all possibilities for the cases of $[6, 4, 6]$ are found among $\mathfrak{s}_{6,158}$ through $\mathfrak{s}_{6,182}$ in Šnobl.

With this in mind we observe a few facts regarding an approach to case splitting $[6, 4, 6]$ and identifying all its cases in Šnobl. To transform $[6, 4, 6]$ into one of $\mathfrak{s}_{6,158}$ through $\mathfrak{s}_{6,182}$, we can change basis in one of the following ways:

i) One can apply any automorphism of the nilradical to the basis. This will leave the nilradical in its current state but will conjugate the matrix $\mathrm{ad}(e_6)$.

ii) One can add to e_6 any combination of $e_1, \ldots, e_5$ and this will not alter the nilradical.

iii) One may scale e_6.

With regards to i), it's well known that the characteristic polynomial is left unchanged by conjugation. Surprisingly, ii) does not alter the characteristic polynomial either. The adjoint representation of e_6 restricted to $e_1, \ldots, e_5$ is

$$\mathrm{ad}(e_6) = \begin{bmatrix} a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c & -d \\ 0 & -b & 0 & -d & -f \\ 0 & 1 & 0 & a & b \\ 0 & 0 & 1 & 0 & a \end{bmatrix}.$$

The characteristic polynomial of $\mathrm{ad}(e_6)$ can be written

$$\chi = (t-a)(t^4 + At^2 + B) \tag{3.3}$$

where

$$A = \left(-1/2\, a^2 + c + f\right)$$

and

$$B = 1/16\, a^4 - 1/4\, a^2 c - 1/4\, a^2 f + abd - b^2 c + cf - d^2.$$

The characteristic polynomial of $\mathrm{ad}(e_6)$ (restricted to $e_1, \ldots, e_5$) for each of the Lie algebras $\mathfrak{s}_{6,158}$ through $\mathfrak{s}_{6,182}$ in Šnobl that correspond to the cases of $[6,4,6]$ will have root patterns given by the root patterns of equation (3.3) which depend on the values of the parameters a, b, c, d, and f. In this way the primary invariant which distinguishes the Lie algebras $\mathfrak{s}_{6,158}$ through $\mathfrak{s}_{6,182}$ is the root pattern of the characteristic polynomial (3.3). We plan to use this approach to identify the Lie algebra $[6,4,6]$ in Šnobl.

3.3 Seven-dimensional Lie algebras on four-dimensional quotients

We investigate pairs $(\mathfrak{g}, \mathfrak{h})$ involving a seven-dimensional Lie algebra $\mathfrak{g}$ with a three-dimensional subalgebra $\mathfrak{h}$ admitting a reductive complement m, where $\mathfrak{g}/\mathfrak{h}$ admits an $\mathrm{ad}(\mathfrak{h})$ invariant inner product having Lorentzian signature. Note that $\mathfrak{h}$ acts as a three-dimensional subalgebra of $\mathfrak{so}(3,1)$. The three-dimensional subalgebras of $\mathfrak{so}(3,1)$ are $F3$, $F4$, $F5$, $F6$, and $F7$, as classified in Winternitz [20]. The results of this section give the database entries $[7,4,1]$, $[7,4,2]$, $[7,4,3]$, $[7,4,4]$, and $[7,4,5]$.

3.3.1 $F3$

The basis in this case is defined by the following matrices

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

designated as e_5, e_6, and e_7 of $\mathfrak{g}$, respectively. Observe that $[e_5, e_6] = e_7$, $[e_5, e_7] = -e_6$, and $[e_6, e_7] = e_5$. Then we have the following structure equations by assuming the above matrices define the adjoint action of e_5, e_6, and e_7 respectively, restricted to a reductive complement $m = \mathrm{span}(\{e_1, e_2, e_3, e_4\})$:

$$\begin{array}{llll} [e_1, e_5] = 0, & [e_2, e_5] = -e_3, & [e_3, e_5] = e_2, & [e_4, e_5] = 0, \\ [e_1, e_6] = e_3, & [e_2, e_6] = 0, & [e_3, e_6] = -e_1, & [e_4, e_6] = 0, \\ [e_1, e_7] = -e_2, & [e_2, e_7] = e_1, & [e_3, e_7] = 0, & [e_4, e_7] = 0, \\ [e_5, e_6] = e_7, & [e_5, e_7] = -e_6, & [e_6, e_7] = e_5. & \end{array}$$

The remaining structure equations to be determined are:

$$[e_1, e_2] = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7,$$

$$[e_1, e_3] = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + b_5 e_5 + b_6 e_6 + b_7 e_7,$$

$$[e_1, e_4] = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5 + c_6 e_6 + c_7 e_7,$$

$$[e_2, e_3] = f_1e_1 + f_2e_2 + f_3e_3 + f_4e_4 + f_5e_5 + f_6e_6 + f_7e_7,$$
$$[e_2, e_4] = g_1e_1 + g_2e_2 + g_3e_3 + g_4e_4 + g_5e_5 + g_6e_6 + g_7e_7,$$
$$[e_3, e_4] = h_1e_1 + h_2e_2 + h_3e_3 + h_4e_4 + h_5e_5 + h_6e_6 + h_7e_7.$$

Upon imposing the Jacobi identities *and extracting and solving only the linear conditions* we reduce the number of unknowns and arrive at the following:

$$[e_1, e_2] = -b_2e_3 - b_6e_7,\ [e_1, e_3] = b_2e_2 + b_6e_6,\ [e_1, e_4] = h_3e_1 + h_7e_5,$$
$$[e_2, e_3] = -b_2e_1 - b_6e_5,\ [e_2, e_4] = h_3e_2 + h_7e_6,\ [e_3, e_4] = h_3e_3 + h_7e_7.$$

Then using Cartan's formula (2.1) we get the conditions $b_2(h_3) - 2h_7 = 0$ and $b_2(h_7) - 2b_6(h_3) = 0$ on the remaining Jacobi identities. Solving these conditions gives two cases (each case thus satisfying all Jacobi identities). Case 1,

$$[e_1, e_2] = -b_2e_3 - b_6e_7,\ [e_1, e_3] = b_2e_2 + b_6e_6,\ [e_1, e_4] = h_3e_1 + h_7e_5,$$
$$[e_2, e_3] = -b_2e_1 - b_6e_5,\ [e_2, e_4] = 0,\ [e_3, e_4] = 0,$$

and Case 2,

$$[e_1, e_2] = -b_2e_3 - \frac{b2^2}{4}e_7,\ [e_1, e_3] = b_2e_2 + \frac{b2^2}{4}e_6,\ [e_1, e_4] = h_3e_1 + \frac{h_3b_2}{2}e_5,$$
$$[e_2, e_3] = -b_2e_1 - \frac{b2^2}{4}e_5,\ [e_2, e_4] = h_3e_2 + \frac{h_3b_2}{2}e_6,\ [e_3, e_4] = h_3e_3 + \frac{h_3b_2}{2}e_7,$$

where, respectively, Case 1 corresponds to the conditions $h_3 = 0, h_7 = 0$ with b_2, b_6 arbitrary, and Case 2 corresponds to $b_6 = \frac{1}{4}b_2^2, h_7 = \frac{1}{2}h_3b_2$ with b_2, h_3 arbitrary.

<u>Case 1:</u> $h_3 = 0, h_7 = 0$ with b_2, b_6 arbitrary. Make the change of basis

$$\{e_1 + \frac{b_2}{2}e_5, e_2 + \frac{b_2}{2}e_6, e_3 + \frac{b_2}{2}e_7, e_4, e_5, e_6, e_7\},$$

and set $a = -b_6 + \frac{1}{4}b_2^2$. The multiplication table then becomes

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	ae_7	$-ae_6$	.	.	e_3	$-e_2$
e_2		.	ae_5	.	$-e_3$	.	e_1
e_3			.	.	e_2	$-e_1$	.
e_4				.	.	.	.
e_5					.	e_7	$-e_6$
e_6						.	e_5
e_7							.

noting the action of the isotropy remains visible. However, for the subcase $a = 0$, the invariant metric admits additional symmetries. The $\mathrm{ad}(\mathfrak{h})$-invariant inner product on $\mathfrak{g}/\mathfrak{h}$ can be seen to be given by

$$g = A\sigma^1 \otimes \sigma^1 + A\sigma^2 \otimes \sigma^2 + A\sigma^3 \otimes \sigma^3 + B\sigma^4 \otimes \sigma^4$$

where A, B are constants. The metric has vanishing curvature tensor when $a = 0$. Therefore we break into subcases $a > 0$ and $a < 0$.

<u>Subcase 1:</u> $a > 0$. The change of basis

$$\left\{e_4, \frac{e_1}{\sqrt{a}}, \frac{e_2}{\sqrt{a}}, \frac{e_3}{\sqrt{a}}, e_5, e_6, e_7\right\}$$

gives the table

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	0	.	0	0	0	0
e_2	0	.	e_7	$-e_6$	.	e_4	$-e_3$
e_3			.	e_5	$-e_4$	.	e_2
e_4				.	e_3	$-e_2$	.
e_5					.	e_7	$-e_6$
e_6						.	e_5
e_7							.

The change of basis

$$\left\{\frac{1}{2}(e_4+e_7), \frac{1}{2}(e_3+e_6), -\frac{1}{2}(e_2+e_5), \frac{1}{2}(e_4-e_7), \frac{1}{2}(e_3-e_6), \frac{1}{2}(e_2-e_5), -e_1\right\}$$

shows the Lie algebra has classification $\mathfrak{so}(3)\oplus\mathfrak{so}(3)\oplus\mathbb{R}$:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	e_3	$-e_2$	.	.	.	.
e_2		.	e_1	.	.	.	.
e_3			.	.	.	.	.
e_4				.	e_6	$-e_5$	.
e_5					.	e_4	.
e_6						.	.
e_7							.

These structure equations are stored as $[7,4,1]$ in the database with isotropy $\{-e_3-e_6, e_2-e_5, e_1-e_4\}$.

<u>Subcase 2:</u> $a<0$. The change of basis

$$\left\{e_4, \frac{e_1}{\sqrt{-a}}, \frac{e_2}{\sqrt{-a}}, \frac{e_3}{\sqrt{-a}}, e_5, e_6, e_7\right\}$$

gives the table

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	0	.	0	0	0	0
e_2	0	.	$-e_7$	e_6	.	e_4	$-e_3$
e_3			.	$-e_5$	$-e_4$	.	e_2
e_4				.	e_3	$-e_2$	.
e_5					.	e_7	$-e_6$
e_6						.	e_5
e_7							.

The change of basis

$$\Big\{e_2 + e_3 + e_4, e_1 - e_5, -e_2 - e_4, -e_1 + e_6, -e_3 - e_4, e_1 - e_5 - e_6, -e_7\Big\}$$

shows the Lie algebra has classification $\mathfrak{so}(3,1) \oplus \mathbb{R}$:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	e_2	e_3	$-e_4$	$-e_5$	.	.
e_2		.	.	$-e_1$	e_6	$-e_3$	.
e_3			.	$-e_6$	$-e_1$	e_2	.
e_4				.	.	$-e_5$	.
e_5					.	e_4	.
e_6						.	.
e_7							.

These structure equations are stored as $[7,4,2]$ with isotropy $\{-e_1-e_3-e_5, -e_4-e_6, e_2-e_6\}$.

<u>Case 2:</u> $b_6 = \frac{1}{4}b_2^2, h_7 = \frac{1}{2}h_3 b_2$ with b_2, h_3 arbitrary. Make the change of basis

$$\{e_1 + \frac{b_2}{2}e_5, e_2 + \frac{b_2}{2}e_6, e_3 + \frac{b_2}{2}e_7, e_4, e_5, e_6, e_7\}$$

The multiplication table then becomes

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	.	.	h_3e_1	.	e_3	$-e_2$
e_2		.	.	h_3e_2	$-e_3$	.	e_1
e_3			.	h_3e_3	e_2	$-e_1$	.
e_4				.	.	.	.
e_5					.	e_7	$-e_6$
e_6						.	e_5
e_7							.

However, for all values of h_3 the invariant metric admits a ten-dimensional isometry algebra. The ad($\mathfrak{h}$)-invariant inner product on $\mathfrak{g}/\mathfrak{h}$ can be seen to be given by

$$g = A\sigma^1 \otimes \sigma^1 + A\sigma^2 \otimes \sigma^2 + A\sigma^3 \otimes \sigma^3 + B\sigma^4 \otimes \sigma^4$$

where A, B are constants, with constant sectional curvature $K = \frac{-h_3^2}{B}$ for $h_3 \neq 0$.

The results of this subsection give the database entries [7, 4, 1] and [7, 4, 2].

3.3.2 $F4$

The basis in this case is defined by the following matrices

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

designated as e_5, e_6, and e_7 of $\mathfrak{g}$, respectively. Observe that $[e_5, e_6] = e_7$, $[e_5, e_7] = -e_6$, and $[e_6, e_7] = -e_5$. Then we have the following structure equations by assuming the above matrices define the adjoint action of e_5, e_6, and e_7 respectively, restricted to a reductive complement $m = \text{span}(\{e_1, e_2, e_3, e_4\})$:

$$\begin{array}{llll} [e_1, e_5] = -e_2, & [e_2, e_5] = e_1, & [e_3, e_5] = 0, & [e_4, e_5] = 0, \\ [e_1, e_6] = -e_4, & [e_2, e_6] = 0, & [e_3, e_6] = 0, & [e_4, e_6] = -e_1, \\ [e_1, e_7] = 0, & [e_2, e_7] = -e_4, & [e_3, e_7] = 0, & [e_4, e_7] = -e_2, \\ [e_5, e_6] = e_7, & [e_5, e_7] = -e_6, & [e_6, e_7] = -e_5. & \end{array}$$

The remaining structure equations are to be determined:

$$[e_1, e_2] = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7,$$

$$[e_1, e_3] = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5 + b_6e_6 + b_7e_7,$$

$$[e_1, e_4] = c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5 + c_6e_6 + c_7e_7,$$

$$[e_2, e_3] = f_1e_1 + f_2e_2 + f_3e_3 + f_4e_4 + f_5e_5 + f_6e_6 + f_7e_7,$$

$$[e_2, e_4] = g_1e_1 + g_2e_2 + g_3e_3 + g_4e_4 + g_5e_5 + g_6e_6 + g_7e_7,$$
$$[e_3, e_4] = h_1e_1 + h_2e_2 + h_3e_3 + h_4e_4 + h_5e_5 + h_6e_6 + h_7e_7.$$

We again impose the Jacobi identities and extract and solve the linear equations first. This gives

$$[e_1, e_2] = c_2e_4 + a_5e_5,\ [e_1, e_3] = -h_4e_1 + b_7e_7,\ [e_1, e_4] = c_2e_2 + a_5e_6,$$
$$[e_2, e_3] = -h_4e_2 - b_7e_6,\ [e_2, e_4] = -c_2e_1 + a_5e_7,\ [e_3, e_4] = b_7e_5 + h_4e_4.$$

Using Cartan's formula (2.1) we find the conditions on the remaining Jacobi identities to be $-c_2(h_4) + 2b_7 = 0$ and $2a_5(h_4) - b_7(c_2) = 0$. Solving this system we arrive at the following two cases, each satisfying the Jacobi identities. Case 1,

$$[e_1, e_2] = c_2e_4 + a_5e_5,\ [e_1, e_3] = 0,\ [e_1, e_4] = c_2e_2 + a_5e_6,$$
$$[e_2, e_3] = 0,\ [e_2, e_4] = -c_2e_1 + a_5e_7,\ [e_3, e_4] = 0,$$

and Case 2,

$$[e_1, e_2] = c_2e_4 + \frac{c_2^2}{4}e_5,\ [e_1, e_3] = -h_4e_1 + \frac{h_4c_2}{2}e_7,\ [e_1, e_4] = c_2e_2 + \frac{c_2^2}{4}e_6,$$
$$[e_2, e_3] = -h_4e_2 - \frac{h_4c_2}{2}e_6,\ [e_2, e_4] = -c_2e_1 + \frac{c_2^2}{4}e_7,\ [e_3, e_4] = h_4e_4 + \frac{h_4c_2}{2}e_5,$$

where, respectively, Case 1 is $\{b_7 = 0, h_4 = 0\}$ with a_5 and c_2 arbitrary, and Case 2 is $\{a_5 = c_2^2/4, b7 = h_4c_2/2\}$ with c_2, b_7 and h_4 arbitrary.

Case 1: $b_7 = 0, h_4 = 0$. Make the change of basis

$$\{e_3, e_1 - \frac{c_2}{2}e_7, e_2 + \frac{c_2}{2}e_6, e_4 + \frac{c_2}{2}e_5, e_5, e_6, e_7\},$$

and set $a = a_5 - \frac{c_2^2}{4}$. The multiplication table then becomes

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	.	.	.	.	.	.
e_2		.	ae_5	ae_6	$-e_3$	$-e_4$	.
e_3			.	ae_7	e_2	.	$-e_4$
e_4				.	.	$-e_2$	$-e_3$
e_5					.	e_7	$-e_6$
e_6						.	$-e_5$
e_7							.

noting the action of the isotropy remained visible. The case $a = 0$ admits a flat invariant metric. The $\mathrm{ad}(\mathfrak{h})$-invariant inner product on $\mathfrak{g}/\mathfrak{h}$ can be seen to be given by

$$g = A\sigma^1 \otimes \sigma^1 + B\sigma^2 \otimes \sigma^2 + B\sigma^3 \otimes \sigma^3 - B\sigma^4 \otimes \sigma^4$$

where A, B are constants. If $a = 0$, the curvature tensor vanishes. Therefore we break into cases for $a > 0$ and $a < 0$.

<u>Subcase 1:</u> $a > 0$. The change of basis

$$\left\{e_1, \frac{e_2}{\sqrt{a}}, \frac{e_3}{\sqrt{a}}, \frac{e_4}{\sqrt{a}}, e_5, e_6, e_7\right\}$$

gives the table

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	.	.	.	.	.	.
e_2		.	e_5	e_6	$-e_3$	$-e_4$	.
e_3			.	e_7	e_2	.	$-e_4$
e_4				.	.	$-e_2$	$-e_3$
e_5					.	e_7	$-e_6$
e_6						.	$-e_5$
e_7							.

The change of basis

$$\left\{e_2, e_3, e_5, -e_4, -e_6, -e_7, e_1\right\}$$

shows the Lie algebra has classification $\mathfrak{so}(3,1) \oplus \mathbb{R}$, the structure equations of which are stored as $[7,4,3]$ in the database. The isotropy is $\{e_3, -e_5, -e_6\}$.

Subcase 2: $a < 0$. The change of basis

$$\left\{e_1, \frac{e_2}{\sqrt{-a}}, \frac{e_3}{\sqrt{-a}}, \frac{e_4}{\sqrt{-a}}, e_5, e_6, e_7\right\}$$

gives the table

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	.	.	.	.	.	.
e_2		.	$-e_5$	$-e_6$	$-e_3$	$-e_4$	.
e_3			.	$-e_7$	e_2	.	$-e_4$
e_4				.	.	$-e_2$	$-e_3$
e_5					.	e_7	$-e_6$
e_6						.	$-e_5$
e_7							.

The change of basis

$$\left\{\frac{1}{2}(e_4 - e_5), \frac{1}{2}(e_2 + e_7), -\frac{1}{2}(e_3 - e_6), \frac{1}{2}(e_4 + e_5), \frac{1}{2}(e_2 - e_7), \frac{1}{2}(e_3 + e_6), e_1\right\}$$

shows the Lie algebra has classification $\mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1) \oplus \mathbb{R}$, the structure equations of which are stored as $[7,4,4]$ in the database. The isotropy is given by $\{-e_1 + e_4, e_3 + e_6, e_2 - e_5\}$.

Case 2: $a_5 = \frac{c_2^2}{4}, b_7 = \frac{h_4 c_2}{2}$. Make the change of basis

$$\{e_1 - \frac{c_2}{2} e_7, e_2 + \frac{c_2}{2} e_6, e_3, e_4 + \frac{c_2}{2} e_5, e_5, e_6, e_7\}$$

and note that the isotropy remains undisturbed. The multiplication table then becomes

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	.	$-h_4e_1$	.	$-e_2$	$-e_4$	.
e_2		.	$-h_4e_2$	.	e_1	.	$-e_4$
e_3			.	h_4e_4	.	.	.
e_4				.	.	$-e_1$	$-e_2$
e_5					.	e_7	$-e_6$
e_6						.	$-e_5$
e_7							.

However, for all values of h_4 the invariant metric admits additional symmetries. Indeed the ad($\mathfrak{h}$)-invariant inner product on $\mathfrak{g}/\mathfrak{h}$ can be seen to be given by

$$g = A\sigma^1 \otimes \sigma^1 + A\sigma^2 \otimes \sigma^2 + B\sigma^3 \otimes \sigma^3 - A\sigma^4 \otimes \sigma^4$$

where A, B are constants. The metric admits a ten-dimensional isometry algebra. If $h_4 \neq 0$, the metric has constant sectional curvature $-1/B$, and if $h_4 = 0$, the curvature tensor vanishes.

The results of this subsection give the database entries [7, 4, 3] and [7, 4, 4].

3.3.3 $F5$

The basis in this case is defined by the following matrices

$$\begin{bmatrix} 0 & -\cos\theta & 0 & 0 \\ \cos\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta \\ 0 & 0 & -\sin\theta & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

designated as e_5, e_6, and e_7 of $\mathfrak{g}$, respectively, where $\theta \in (0, \pi/2)$. Observe that $[e_5, e_6] = \sin\theta e_6 + \cos\theta e_7$, $[e_5, e_7] = -\cos\theta e_6 + \sin\theta e_7$, and $[e_6, e_7] = 0$. Then we have the following structure equations by assuming the above matrices define the adjoint action of e_5, e_6, and e_7 respectively, restricted to a reductive complement $m = \text{span}(\{e_1, e_2, e_3, e_4\})$:

$$
\begin{array}{lll}
[e_1, e_5] = -\cos\theta e_2, & [e_2, e_5] = \cos\theta e_1, & [e_3, e_5] = \sin\theta e_4, \\
[e_1, e_6] = -e_3 + e_4, & [e_2, e_6] = 0, & [e_3, e_6] = e_1, \\
[e_1, e_7] = 0, & [e_2, e_7] = -e_3 + e_4, & [e_3, e_7] = e_2, \\
[e_4, e_5] = \sin\theta e_3, & [e_4, e_6] = e_1, & [e_4, e_7] = e_2, \\
[e_5, e_6] = \sin\theta e_6 + \cos\theta e_7, & [e_5, e_7] = -\cos\theta e_6 + \sin\theta e_7, & [e_6, e_7] = 0.
\end{array}
$$

The remaining structure equations are to be determined:

$$
\begin{aligned}
[e_1, e_2] &= a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7, \\
[e_1, e_3] &= b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + b_5 e_5 + b_6 e_6 + b_7 e_7, \\
[e_1, e_4] &= c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5 + c_6 e_6 + c_7 e_7, \\
[e_2, e_3] &= f_1 e_1 + f_2 e_2 + f_3 e_3 + f_4 e_4 + f_5 e_5 + f_6 e_6 + f_7 e_7, \\
[e_2, e_4] &= g_1 e_1 + g_2 e_2 + g_3 e_3 + g_4 e_4 + g_5 e_5 + g_6 e_6 + g_7 e_7, \\
[e_3, e_4] &= h_1 e_1 + h_2 e_2 + h_3 e_3 + h_4 e_4 + h_5 e_5 + h_6 e_6 + h_7 e_7.
\end{aligned}
$$

Imposing the Jacobi identities reveals the only possibility for all Jacobi identities to be satisfied is for all parameters a_i, b_i, c_i, f_i, g_i, h_i to vanish. The multiplication table that follows is:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	.	.	.	$-\cos\theta\ e_2$	$-e_3 + e_4$	.
e_2		.	.	.	$\cos\theta\ e_1$	.	$-e_3 + e_4$
e_3			.	.	$\sin\theta\ e_4$	e_1	e_2
e_4				.	$\sin\theta\ e_3$	e_1	e_2
e_5					.	$\sin\theta\ e_6 + \cos\theta\ e_7$	$-\cos\theta\ e_6 + \sin\theta\ e_7$
e_6						.	.
e_7							.

However, the invariant metric in this case is flat. Indeed the ad($\mathfrak{h}$)-invariant inner product on $\mathfrak{g}/\mathfrak{h}$ can be seen to be given by

$$g = A\sigma^1 \otimes \sigma^1 + A\sigma^2 \otimes \sigma^2 + A\sigma^3 \otimes \sigma^3 - A\sigma^4 \otimes \sigma^4$$

where A is a constant, and has vanishing curvature tensor.

Therefore this section contributes no entries to the database.

3.3.4 $F6$

The basis in this case is defined by the following matrices

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

designated as e_5, e_6, and e_7 of $\mathfrak{g}$, respectively. Observe that $[e_5, e_6] = 2e_7$, $[e_5, e_7] = -2e_6$, and $[e_6, e_7] = 0$. Then we have the following structure equations by assuming the above matrices define the adjoint action of e_5, e_6, and e_7 respectively, restricted to a reductive complement $m = \text{span}(\{e_1, e_2, e_3, e_4\})$:

$$\begin{aligned}
&[e_1, e_5] = -2e_2, && [e_2, e_5] = 2e_1, && [e_3, e_5] = 0, && [e_4, e_5] = 0, \\
&[e_1, e_6] = -e_3 + e_4, && [e_2, e_6] = 0, && [e_3, e_6] = 0, && [e_4, e_6] = e_1, \\
&[e_1, e_7] = 0, && [e_2, e_7] = -e_3 + e_4, && [e_3, e_7] = 0, && [e_4, e_7] = e_2, \\
&[e_5, e_6] = 2e_7, && [e_5, e_7] = -2e_6, && [e_6, e_7] = 0.
\end{aligned}$$

The remaining structure equations are to be determined:

$$\begin{aligned}
[e_1, e_2] &= a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7, \\
[e_1, e_3] &= b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5 + b_6e_6 + b_7e_7, \\
[e_1, e_4] &= c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5 + c_6e_6 + c_7e_7, \\
[e_2, e_3] &= f_1e_1 + f_2e_2 + f_3e_3 + f_4e_4 + f_5e_5 + f_6e_6 + f_7e_7, \\
[e_2, e_4] &= g_1e_1 + g_2e_2 + g_3e_3 + g_4e_4 + g_5e_5 + g_6e_6 + g_7e_7, \\
[e_3, e_4] &= h_1e_1 + h_2e_2 + h_3e_3 + h_4e_4 + h_5e_5 + h_6e_6 + h_7e_7.
\end{aligned}$$

Upon imposing the Jacobi identities, we reduce the number of unknowns and arrive at the following:

$$[e_1,e_2]=-b_2e_3+b_2e_4,\ [e_1,e_3]=-h_4e_1+b_2e_2+g_7e_6-\frac{b_2h_4}{2}e_7,$$
$$[e_1,e_4]=-h_4e_1+b_2e_2+g_7e_6-\frac{b_2h_4}{2}e_7,\ [e_2,e_3]=-b_2e_1-h_4e_2+\frac{b_2h_4}{2}e_6+g_7e_7,$$
$$[e_2,e_4]=-b_2e_1-h_4e_2+\frac{b_2h_4}{2}e_6+g_7e_7,\ [e_3,e_4]=-e_3h_4+e_4h_4,$$

noting all Jacobi identities are satisfied. The algebra is solvable and indecomposable for generic parameter values. The nilradical is $\{e1,e2,e3-e4,e6,e7\}$. If we change basis by

$$\{e_3-e_4,e_3-e_4+e_6,-e_2+e_3-e_4-e_7,e_1+e_3-e_4+b_2e_7,e_7,e_3,e_5\}$$

we see the corresponding abstract multiplication table for the nilradical is

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	.	e_1	.
e_3			.	.	e_1
e_4				.	.
e_5					.

The structure of the nilradical is precisely the structure of $\mathfrak{n}_{5,3}$ in Šnobl [11]. The remaining structure equations are given by the following table where $a=b_2,\ b=h_4$, and $c=g7$:

	e_6	e_7
e_1	$-be_1$	.
e_2	$(1-b)e_1-e_4+ae_5$	$-2e_5$
e_3	$Ae_1-\frac{1}{2}abe_2-(b+1)e_3+ae_4+Be_5$	$4e_1-2e_2-2e_4+2ae_5$
e_4	$-ce_1+ce_2-be_4+\frac{1}{2}abe_5$	$-(2+2a)e_1+2ae_2+2e_3+2e_5$
e_5	$-e_1+e_3+e_5$	$-2e_1+2e_2$
e_6	.	.
e_7		.

with $A = (1 - a + \frac{1}{2}ab)$, $B = -(a^2 + c + b + 1)$. *These structure equations are stored as* $[7,4,5]$ *in the database created in Chapter* 4. Attempts to separate all cases of $[7,4,5]$ will be a future project. It's important to note that there is no classification of solvable seven-dimensional Lie algebras. However, we may observe that the vectors $[e_1, e_2, e_3, e_4, e_5, e_7]$ form a subalgebra. If $a \neq 0$, then the change of basis

$$[\frac{e_1}{a}, -\frac{(a+1)\,e_1}{a} + e_2 + \frac{e_3}{a} + \frac{e_5}{a}, -\frac{e_1}{a} + \frac{e_3}{a} + \frac{e_5}{a}, -\frac{e_1}{a} + \frac{e_4}{a}, \frac{e_1}{a} - \frac{e_4}{a} + e_5, \frac{1}{2}\,e_7]$$

gives the following table (which is precisely $\mathfrak{s}_{6,166}$ with $\alpha = -1$):

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	.	.	.	.
e_2		.	.	e_1	.	$-e_4$
e_3			.	.	e_1	e_5
e_4				.	.	e_2
e_5					.	$-e_3$
e_6						.

If $a = 0$, the change of basis

$$[-e_1, -\frac{3}{2}\,e_1 + e_2 + \frac{1}{2}\,e_3 + e_5, \frac{1}{2}\,e_1 - e_2 + \frac{1}{2}\,e_3 + e_5, \frac{1}{2}\,e_2 - \frac{1}{2}\,e_4 - e_5,$$
$$e_1 - \frac{1}{2}\,e_2 + \frac{1}{2}\,e_4 - e_5, -e_1 + \frac{1}{2}\,e_2 - \frac{1}{2}\,e_4 + e_5 - \frac{1}{2}\,e_7]$$

gives the same table. Thus for generic a, the subalgebra is $\mathfrak{s}_{6,166}$ with $\alpha = -1$ in Šnobl [11].

3.3.5 *F*7

The basis in this case is defined by the following matrices

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

designated as e_5, e_6, and e_7 of $\mathfrak{g}$, respectively. Observe that $[e_5, e_6] = 2e_6$, $[e_5, e_7] = 2e_7$, and $[e_6, e_7] = 0$. Then we have the following structure equations by assuming the above matrices define the adjoint action of e_5, e_6, and e_7 respectively, restricted to a reductive complement $m = \text{span}(\{e_1, e_2, e_3, e_4\})$:

$$\begin{array}{llll}
[e_1, e_5] = 0, & [e_2, e_5] = 0, & [e_3, e_5] = 2e_4, & [e_4, e_5] = 2e_3, \\
[e_1, e_6] = -e_3 + e_4, & [e_2, e_6] = 0, & [e_3, e_6] = e_1, & [e_4, e_6] = e_1, \\
[e_1, e_7] = 0, & [e_2, e_7] = -e_3 + e_4, & [e_3, e_7] = e_2, & [e_4, e_7] = e_2, \\
[e_5, e_6] = 2e_6, & [e_5, e_7] = 2e_7, & [e_6, e_7] = 0. &
\end{array}$$

The remaining structure equations to be determined are:

$$\begin{aligned}
[e_1, e_2] &= a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7, \\
[e_1, e_3] &= b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5 + b_6e_6 + b_7e_7, \\
[e_1, e_4] &= c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5 + c_6e_6 + c_7e_7, \\
[e_2, e_3] &= f_1e_1 + f_2e_2 + f_3e_3 + f_4e_4 + f_5e_5 + f_6e_6 + f_7e_7, \\
[e_2, e_4] &= g_1e_1 + g_2e_2 + g_3e_3 + g_4e_4 + g_5e_5 + g_6e_6 + g_7e_7, \\
[e_3, e_4] &= h_1e_1 + h_2e_2 + h_3e_3 + h_4e_4 + h_5e_5 + h_6e_6 + h_7e_7.
\end{aligned}$$

Imposing the Jacobi identities reveals the only possibility for all Jacobi identities to be satisfied is all parameters a_i, b_i, c_i, f_i, g_i, h_i vanishing. The multiplication table that follows is:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	.	.	.	.	$-e_3 + e_4$	.
e_2		.	.	.	.	.	$-e_3 + e_4$
e_3			.	.	$2e_4$	e_1	e_2
e_4				.	$2e_3$	e_1	e_2
e_5					.	$2e_6$	$2e_7$
e_6						.	.
e_7							.

However, the invariant metric in this case is flat. Note the ad($\mathfrak{h}$)-invariant inner product on $\mathfrak{g}/\mathfrak{h}$ can be seen to be given by

$$g = A\sigma^1 \otimes \sigma^1 + A\sigma^2 \otimes \sigma^2 + A\sigma^3 \otimes \sigma^3 - A\sigma^4 \otimes \sigma^4$$

where A is a constant, and has vanishing curvature tensor.

Therefore this subsection contributes no entries to the database.

3.4 Conclusion

This concludes the use of the Schmidt method in identifying all possible Lorentzian pairs. The new results of this chapter yielded the pairs labeled $[6, 3, 1]$, $[6, 3, 2]$, $[6, 3, 3]$, $[6, 3, 4]$, $[6, 3, 5]$, and $[6, 3, 6]$ for the case of six-dimensional Lie algebra and three-dimensional isotropy; $[6, 4, 1]$, $[6, 4, 2]$, $[6, 4, 3]$, $[6, 4, 4]$, $[6, 4, 5]$, and $[6, 4, 6]$ for the case of six-dimensional Lie algebra and two-dimensional isotropy; $[7, 4, 1]$, $[7, 4, 2]$, $[7, 4, 3]$, $[7, 4, 4]$, and $[7, 4, 5]$ for the case of seven-dimensional Lie algebra and three-dimensional isotropy. The full results for all cases are presented in Appendix A.1.

CHAPTER 4

SOFTWARE FOR CLASSIFICATION OF LORENTZIAN PAIRS

To make available and usable the classification of Lorentzian Lie algebra-subalgebra pairs from Chapter 3, the data structures defining them have been put into a database. By computing enough Lie theoretic invariants for each entry, the pairs are distinguished one from another. Computing those same invariants and making comparisons against the database allows one to classify any Lorentzian pair. The powerful DifferentialGeometry package in Maple developed by Ian Anderson at Utah State University can be used to quickly compute the Lie invariants. Software built on the DifferentialGeometry package has been written for this dissertation to automate for researchers the classification of Lorentzian pairs and allow easy access to the database. We name this new classification software *HomogeneousSpaceClassifier*.

We discuss briefly the Lie invariants used by *HomogeneousSpaceClassifier* and its accompanying database. The code for the classifier and its database can be found at Digital Commons at Utah State University for public download and use. We will then walk through an example demonstrating explicit use of the software in Maple.

The Lorentzian pairs forming the entries in the database are labeled precisely as they were in Chapter 3 and are of the form $[a, b, c]$, where $a = \dim(\mathfrak{g})$, $b = \dim(\mathfrak{g}/\mathfrak{h})$, and for fixed a and b, c simply labels the entries, beginning with 1. Recall that the non-reductive entries have integers $c < 0$. The classifier takes as input a Lie algebra and a basis of vectors defining the isotropy subalgebra. It returns a reference $[a, b, c]$ to the database. Details of its operation are in Section 4.3.

4.1 Methodology and invariants

Given the Lie algebra-subalgebra pair, there are several Lie theoretic invariants computed at different steps of the classification. We list here those invariants used in the classifier. These invariants will be computed first for the Lie algebra and isotropy. Then, quotient algebras will be defined and initialized using ideals from the derived, lower, and upper series when they exist. The same invariants computed for the Lie algebra will then be computed for these new quotient Lie algebras.

It should be noted that there are a few exceptional cases that require additional invariants for classification that are not computed for the rest of the database pairs to minimize storage requirements. These cases will also be discussed. The following invariants are stored in the database for each Lie algebra and quotient Lie algebra.

Algebra Type: The first invariant determined is whether the Lie algebra is abelian, nilpotent, solvable, semi-simple, or generic. Of course an abelian or nilpotent Lie algebra is solvable, but for the purposes of classification we will label those Lie algebras that are solvable but *not* abelian or nilpotent as the solvable Lie algebras. Similarly, if a Lie algebra is non-abelian but nilpotent it will be called nilpotent. Any abelian Lie algebra is simply labeled abelian. Generic Lie algebras are not abelian, nilpotent, solvable, or semi-simple.

Indecomposable: We determine whether or not a Lie algebra is indecomposable. As a separate invariant for a given algebra, we also determine if the nilradical is indecomposable. We store the value $true$ or $false$.

Derived Series Dimensions: Recall that the derived series for a Lie algebra $\mathfrak{g}$ is the sequence $\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \mathfrak{g}^{(2)} \supseteq \cdots$. The dimensions of each ideal in the sequence is stored in a list. If the sequence is solvable then the derived series dimensions will look something like $[5, 3, 1, 0]$, where 5 is the dimension of the Lie algebra, 3 the dimension of the derived algebra, and so on. In the non-solvable cases clearly the sequence does not terminate and the derived series dimensions will be denoted in the form $[5, 3, 3]$, it being understood that the Lie algebra is dimension 5, the derived is dimension 3, and the three's repeat. That is, $3 = \dim(\mathfrak{g}^{(1)}) = \dim(\mathfrak{g}^{(2)}) = \cdots$ and there is no k such that $\mathfrak{g}^{(k)} = 0$ in this example.

Similar dimension counting for the lower and upper series is not performed nor stored in the database as it became apparent that these invariants changed whenever the derived series dimensions changed, for a great many instances at least. Meaning for most pairs the lower and upper series dimension counting wasn't useful for the classifier.

Other Dimensions: Also computed are the dimensions of various Lie algebras and ideals associated with the Lie algebra supplied to the classifier:

- dimension of the nilradical

- dimension of $\mathcal{D}$, the Lie algebra of derivations
- dimension of the Lie algebra of derivations of $\mathcal{D}$ $(*)$
- dimensions of the decomposition of the radical of $\mathcal{D}$

To compute most of these dimensions we must first compute the actual Lie algebras, ideals, or decompositions. Note for $(*)$ that first we compute the derivations of the original abstract Lie algebra. Then, we set up the abstract Lie algebra of those derivations and compute the derivations of this new Lie algebra and count the dimension. This turned out to be very useful and is a quick computation.

Isotropy Data: Needed are a few facts regarding the isotropy subalgebra of a Lorentzian pair.

i) We calculate a general complementary basis $\mathfrak{m}$ to the isotropy subalgebra $\mathfrak{h}$ and determine if $(\mathfrak{m}, \mathfrak{h})$ forms a reductive pair (meaning $[\mathfrak{m}, \mathfrak{h}] \subset \mathfrak{m}$), storing the value $true$ or $false$.
ii) If $true$, then it makes sense to determine if they are a symmetric pair as well (meaning $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$), again storing a $true$ or $false$ in memory.
iii) Also determined is the isotropy type by computing the matrices defining the adjoint action of the isotropy. The abstract Lie algebra defined by these matrices sits as a subalgebra of $\mathfrak{so}(3, 1)$, all subalgebras of which have been classified (up to conjugation) and labeled $F1$ through $F15$, with $F15$ the trivial subalgebra (see Chapter 2 and Winternitz [20]). The command IsotropyType of the DifferentialGeometry package utilizes this classification and is called upon by the classifier for this invariant.

When dealing with generic Lie algebras with one-dimensional isotropy, we determine if the isotropy subalgebra is contained in the semi-simple part of the Levi decomposition, storing $true$ or $false$ or when not applicable, "NA". This is not useful for generic cases with isotropy of higher dimension than one as the Levi-decomposition is a semi-direct sum.

Exceptional Cases: If the classifier is given a three-dimensional semisimple Lie algebra, then it is either $\mathfrak{so}(3)$ or $\mathfrak{sl}(2)$. By computing the Killing form and determining whether it is negative definite, in which case the algebra is $\mathfrak{so}(3)$, or indefinite, giving $\mathfrak{sl}(2)$, we can quickly classify the Lie algebra. This invariant is stored and computed only for instances that a three-dimensional semisimple Lie algebra occurs, as can be the case when we compute quotient algebras created from the ideals in the derived, lower, and upper series of the Lie algebra.

A related scenario is in attempting to classify the two six-dimensional Lie algebras $\mathfrak{so}(3)\oplus\mathfrak{sl}(2)$ and $\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$, which are $[6,4,1]$ and $[6,4,2]$ respectively. The same isotropy type, $F9$, describes both isotropy subalgebras. To distinguish between these Lie algebras, the decomposition must be computed. There is no way to tell a priori which factor comes first in the direct sum. Then using the definiteness of the Killing forms of the factors the classification can quickly be determined. Only these two algebras require this attention in the classifier.

Steps To Classification: Once the classifier is initiated, it will create a list L_1 of all database entries having the same dimension as the user supplied Lie algebra. Step 1 in the classifier is to compute invariants for the Lie algebra in the following order:

i) algebra type,
ii) the derived series dimensions,
iii) indecomposability,
iv) dimension of the nilradical,
v) dimension of $\mathcal{D}$, the Lie algebra of derivations,
vi) dimension of the Lie algebra of derivations of $\mathcal{D}$,
vii) dimensions of the decomposition of the radical of $\mathcal{D}$,
viii) dimensions of the decomposition of the Lie algebra.

From these properties a comparison is made to the internal database of pairs. All pairs in L_1 without identical properties are removed thereby creating a new list L_2 for more processing. Step 2 repeats this process for the derived algebra $g^{(1)}$, then for $g^{(2)}$ and so on. This step is not performed for Abelian or semisimple Lie algebras. For generic Lie algebras, the derived series will at some stage repeat ad infinitum. The classifier simply detects the k for which $g^{(k)}$ is repeated and computes invariants for $g^{(j)}$ for $j \leq k$.

Similarly, Step 3 repeats for non-Abelian, non-semisimple Lie algebras the process on $g_{(1)}$, then $g_{(2)}$, and so on, of the lower series and likewise for ideals in the upper series. Step

4 classifies by orbit dimension and Step 5 by the three properties of the isotropy subalgebra described above.

4.1.1 Database entry

The following is the first entry in the database giving an idea of how the properties above are stored. The appendices contain the entire database and code for the classifier. Note the entry "PetrovClassification" refers to a vector field system in Petrov [2] giving this Lie algebra-subalgebra pair. This will be discussed further in Chapter 5.

```
####################################################################
#       [3, 2, 1]
####################################################################
DGTable[[3, 2, 1]] := table();
DGTable[[3, 2, 1]]["Parameters"] := [[], [[]]]:
DGTable[[3, 2, 1]]["Remarks"] := []:
DGTable[[3, 2, 1]]["Issues"] := []:
DGTable[[3, 2, 1]]["Reference"] := "R(3,1), Bowers":
# Defining Structure
DGTable[[3, 2, 1]]["StructureConstants"] := [[[1, 3, 2], -1], [[2, 3, 1], 1]]:
DGTable[[3, 2, 1]]["Isotropy"] := [[0, 0, 1]]:
DGTable[[3, 2, 1]]["Complement"] := [[1, 0, 0], [0, 1, 0]]:
DGTable[[3, 2, 1]]["PetrovClassification"] := [[30, 1, 0]]:
# Algebra Properties
DGTable[[3, 2, 1]]["AlgebraType"] := "Solvable":
DGTable[[3, 2, 1]]["Indecomposable"] := true:
DGTable[[3, 2, 1]]["DecompositionDimensions"] := [3]:
DGTable[[3, 2, 1]]["DerivedSeriesDimensions"] := [3, 2, 0]:
DGTable[[3, 2, 1]]["NilradicalDimension"] := 2:
DGTable[[3, 2, 1]]["NilradicalIndecomposable"] := false:
DGTable[[3, 2, 1]]["DerivationsDimension"] := 4:
DGTable[[3, 2, 1]]["DecompositionOfRadicalOfDerivationsDimensions"] := [4]:
DGTable[[3, 2, 1]]["DerivationsOfDerivationsDimension"] := 4:
# Isotropy Subalgebra Properties
DGTable[[3, 2, 1]]["IsotropyType"] := "F12":
DGTable[[3, 2, 1]]["IsotropySubalgebraReductivePair"] := true:
DGTable[[3, 2, 1]]["IsotropySubalgebraSymmetricPair"] := true:
DGTable[[3, 2, 1]]["IsotropySubalgebraInSemisimplePart"] := "NA":
# Derived 1 Algebra Properties
DGTable[[3, 2, 1]]["Derived1_AlgebraType"] := "Abelian":
DGTable[[3, 2, 1]]["Derived1_Indecomposable"] := false:
```

```
DGTable[[3, 2, 1]]["Derived1_DecompositionDimensions"] := []:
DGTable[[3, 2, 1]]["Derived1_DerivedSeriesDimensions"] := [2, 0]:
DGTable[[3, 2, 1]]["Derived1_NilradicalDimension"] := 2:
DGTable[[3, 2, 1]]["Derived1_NilradicalIndecomposable"] := false:
DGTable[[3, 2, 1]]["Derived1_DerivationsDimension"] := 4:
DGTable[[3, 2, 1]]["Derived1_DecompositionOfRadicalOfDerivationsDimensions"] := []:
DGTable[[3, 2, 1]]["Derived1_DerivationsOfDerivationsDimension"] := 4:
# Derived 1 Quotient Algebra Properties
DGTable[[3, 2, 1]]["Derived1_Quotient_AlgebraType"] := "Abelian":
DGTable[[3, 2, 1]]["Derived1_Quotient_Indecomposable"] := true:
DGTable[[3, 2, 1]]["Derived1_Quotient_DecompositionDimensions"] := []:
DGTable[[3, 2, 1]]["Derived1_Quotient_DerivedSeriesDimensions"] := [1, 0]:
DGTable[[3, 2, 1]]["Derived1_Quotient_NilradicalDimension"] := 1:
DGTable[[3, 2, 1]]["Derived1_Quotient_NilradicalIndecomposable"] := true:
DGTable[[3, 2, 1]]["Derived1_Quotient_DerivationsDimension"] := 1:
DGTable[[3, 2, 1]]["Derived1_Quotient_DecompositionOfRadicalOfDerivationsDimensions"]
     := []:
DGTable[[3, 2, 1]]["Derived1_Quotient_DerivationsOfDerivationsDimension"] := 1:
# Lower 1 Algebra Properties
DGTable[[3, 2, 1]]["Lower1_AlgebraType"] := "Abelian":
DGTable[[3, 2, 1]]["Lower1_Indecomposable"] := false:
DGTable[[3, 2, 1]]["Lower1_DecompositionDimensions"] := []:
DGTable[[3, 2, 1]]["Lower1_DerivedSeriesDimensions"] := [2, 0]:
DGTable[[3, 2, 1]]["Lower1_NilradicalDimension"] := 2:
DGTable[[3, 2, 1]]["Lower1_NilradicalIndecomposable"] := false:
DGTable[[3, 2, 1]]["Lower1_DerivationsDimension"] := 4:
DGTable[[3, 2, 1]]["Lower1_DecompositionOfRadicalOfDerivationsDimensions"] := []:
DGTable[[3, 2, 1]]["Lower1_DerivationsOfDerivationsDimension"] := 4:
# Lower 1 Quotient Algebra Properties
DGTable[[3, 2, 1]]["Lower1_Quotient_AlgebraType"] := "Abelian":
DGTable[[3, 2, 1]]["Lower1_Quotient_Indecomposable"] := true:
DGTable[[3, 2, 1]]["Lower1_Quotient_DecompositionDimensions"] := []:
DGTable[[3, 2, 1]]["Lower1_Quotient_DerivedSeriesDimensions"] := [1, 0]:
DGTable[[3, 2, 1]]["Lower1_Quotient_NilradicalDimension"] := 1:
DGTable[[3, 2, 1]]["Lower1_Quotient_NilradicalIndecomposable"] := true:
DGTable[[3, 2, 1]]["Lower1_Quotient_DerivationsDimension"] := 1:
DGTable[[3, 2, 1]]["Lower1_Quotient_DecompositionOfRadicalOfDerivationsDimensions"]
    := []:
DGTable[[3, 2, 1]]["Lower1_Quotient_DerivationsOfDerivationsDimension"] := 1:
# Lower 2 Algebra Properties
DGTable[[3, 2, 1]]["Lower2_AlgebraType"] := "Abelian":
DGTable[[3, 2, 1]]["Lower2_Indecomposable"] := false:
DGTable[[3, 2, 1]]["Lower2_DecompositionDimensions"] := []:
DGTable[[3, 2, 1]]["Lower2_DerivedSeriesDimensions"] := [2, 0]:
```

```
DGTable[[3, 2, 1]]["Lower2_NilradicalDimension"] := 2:
DGTable[[3, 2, 1]]["Lower2_NilradicalIndecomposable"] := false:
DGTable[[3, 2, 1]]["Lower2_DerivationsDimension"] := 4:
DGTable[[3, 2, 1]]["Lower2_DecompositionOfRadicalOfDerivationsDimensions"] := []:
DGTable[[3, 2, 1]]["Lower2_DerivationsOfDerivationsDimension"] := 4:
# Lower 2 Quotient Algebra Properties
DGTable[[3, 2, 1]]["Lower2_Quotient_AlgebraType"] := "Abelian":
DGTable[[3, 2, 1]]["Lower2_Quotient_Indecomposable"] := true:
DGTable[[3, 2, 1]]["Lower2_Quotient_DecompositionDimensions"] := []:
DGTable[[3, 2, 1]]["Lower2_Quotient_DerivedSeriesDimensions"] := [1, 0]:
DGTable[[3, 2, 1]]["Lower2_Quotient_NilradicalDimension"] := 1:
DGTable[[3, 2, 1]]["Lower2_Quotient_NilradicalIndecomposable"] := true:
DGTable[[3, 2, 1]]["Lower2_Quotient_DerivationsDimension"] := 1:
DGTable[[3, 2, 1]]["Lower2_Quotient_DecompositionOfRadicalOfDerivationsDimensions"]
    := []:
DGTable[[3, 2, 1]]["Lower2_Quotient_DerivationsOfDerivationsDimension"] := 1:
```

4.2 Finding isomorphisms for Lorentzian pairs

To determine if two Lie algebra-subalgebra pairs $(\mathfrak{g}, \mathfrak{h})$ and $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ are equivalent, an isomorphism $\phi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ such that $\phi(\mathfrak{h}) = \tilde{\mathfrak{h}}$ is required. Given a Lie algebra isomorphism $\phi : \mathfrak{g} \to \tilde{\mathfrak{g}}$, then

$$\phi(e_i) = \tilde{e}_i = A_i^j e_j$$

for some matrix A, with $[e_i, e_j] = C_{ij}^k e_k$, and $[\tilde{e}_i, \tilde{e}_j] = \tilde{C}_{ij}^k \tilde{e}_k$. Then observe

$$\begin{aligned}
\phi([e_i, e_j]) &= [\phi(e_i), \phi(e_j)] \\
&= [A_i^l e_l, A_j^m e_m] \\
&= A_i^l A_j^m [e_l, e_m] \\
&= A_i^l A_j^m C_{lm}^p e_p.
\end{aligned}$$

But we also have

$$\begin{aligned}
\phi([e_i, e_j]) &= [\tilde{e}_i, \tilde{e}_j] \\
&= \tilde{C}_{ij}^k \tilde{e}_k \\
&= \tilde{C}_{ij}^k A_k^p e_p.
\end{aligned}$$

Thus to find an isomorphism ϕ, we must solve the $\frac{n^2(n-1)}{2}$ quadratic equations

$$A_i^l A_j^m C_{lm}^k - \tilde{C}_{ij}^k A_k^p = 0, \quad \det\, A \neq 0$$

for A_i^j. However, we must also ensure $\phi(\mathfrak{h}) = \tilde{\mathfrak{h}}$. Recall the *annihilator* of a set $S \subset \mathfrak{g}$ is the subspace ann(S) $\subset \mathfrak{g}^*$, with

$$\mathrm{ann(S)} = \{\theta \in \mathfrak{g}^* \mid \theta(\mathrm{s}) = 0, \text{ for all s} \in \mathrm{S}\},$$

where $\mathfrak{g}^*$ is the dual space to $\mathfrak{g}$. Compute ann($\tilde{\mathfrak{h}}$) and set $\tilde{\theta}(\phi(h)) = 0$, for all $h \in \mathfrak{h}$ and $\tilde{\theta} \in \mathrm{ann}(\tilde{\mathfrak{h}})$. Then if $h = B^k e_k$, we have $\phi(h) = \phi(B^k e_k) = B^\phi(e_k) = B^k A_k^j e_j$, giving $\tilde{\theta}(\phi(h)) = B^k A_k^j e_j$. Then the full set of equations to determine a transformation ϕ which defines the equivalence of two Lie algebra-subalgebra pairs is the following:

$$\begin{cases} A_i^l A_j^m C_{lm}^k - \tilde{C}_{ij}^k A_k^p = 0, \\ B^k A_k^j = 0 \\ \det\, A \neq 0 \end{cases}$$

noting that C_{lm}^k, $\tilde{C}_{ij}^k$ and B^k are known.

The isomorphism equations are in general difficult to solve, even in a computer algebra system. The Lie algebra-subalgebra pair problem is easier as there is a linear system (namely $B^k A_k^j = 0$) which simplifies the quadratic equations of the isomorphism problem. Software solving these systems for the Lie algebra-subalgebra pair problem has been written for this dissertation and is called $IsomorphismForLiePairs$. Its code can be found as a supplemental file to the digital copy of the dissertation and its use is demonstrated in the next section.

4.3 The software in use

As mentioned previously, the classifier is named HomogeneousSpaceClassifier. The pairs in the database can be accessed through the classifier in the following manner. For more information using the DifferentialGeometry package, one may see the help pages.

Some familiarity with basic Maple syntax is assumed.

If we want to initialize the entry $[3,2,1]$ for example, we may enter the following command,

$> LD1, P1 := HomogeneousSpaceClassifier(retrieve = [[3,2,1], alg1]);$

$$LD1, P1 := [e1, e2] = 0,\ [e1, e3] = -e2,\ [e2, e3] = e1, [alg1, [e3]]$$

Note the structure equations, followed by a list containing the name $alg1$ of the Lie algebra $\mathfrak{g}$ and a list of vectors, $[e3]$ in this case, giving a basis for the isotropy subalgebra $\mathfrak{h}$, are output by the program. The name of the algebra is chosen by the user. Also chosen was $LD1$ to name the structure equations, and $P1$ to name the Lorentzian Lie algebra-subalgebra pair $[alg1, [e3]]$ representing $(\mathfrak{g}, \mathfrak{h})$.

The structure equations LD may be initialized for further use by the command $DGsetup$ of the DifferentialGeometry package.

$> DGsetup(LD);$

$$Lie\ algebra : alg1$$

Once initialized, we may work with the Lie algebra. If we supply the pair to the classifier, it will return a list containing $[3,2,1]$:

$> HomogeneousSpaceClassifier(P1);$

$$[[3,2,1]]$$

Using the invertible matrix A,

$> A := Matrix([[-1,0,2],[1,1/2,-1],[-1,1,2]]);$

$$\begin{bmatrix} -1 & 1/2 & 2 \\ 1 & 1/2 & -1 \\ -1 & 1 & 2 \end{bmatrix}$$

we can change basis and initialize new structure equations for $P1$. We may define a linear transformation $\phi : \mathfrak{g} \to \mathfrak{g}$ given by A.

$> phi := LinearTransformation(alg1, alg1, A);$

$$\phi := e1 \to -e1 + e2 - e3,\ e2 \to \frac{1}{2}e1 + \frac{1}{2}e2 + e3,\ e3 \to 2e1 - e2 + 2e3$$

Using ϕ, we generate the new basis and initialize it as a "new" Lie algebra $\tilde{\mathfrak{g}}$.

$> NB1 := ApplyLinearTransformation(phi, [e1, e2, e3]);$

$$NB1 := [-e1 + e2 - e3, \frac{1}{2}e1 + \frac{1}{2}e2 + e3, 2e1 - e2 + 2e3]$$

$> LD2 := LieAlgebraData(NB1, alg2); DGsetup(LD2) :$

$$LDn := [e1, e2] = 7e1 - 3e2 + 5e3,\ [e1, e3] = 4e1 - 2e2 + 3e3,$$

$[e2, e3] = 10e1 - 4e2 + 7e3$

The basis vectors in this output belong to $alg2$ and after having initialized the structure equations, Maple is now active in that frame of reference. We can find the isotropy subalgebra in this new basis a number of ways. An intuitive method may be to compute the components of $\mathfrak{h}$ in the basis

$$[-e1 + e2 - e3, \frac{1}{2}e1 + \frac{1}{2}e2 + e3, 2e1 - e2 + 2e3].$$

This is easily done using the command $GetComponents$.

$> GC := GetComponents(P1[2], NB1);$

$$GC := [[-3, 2, -2]]$$

Then the isotropy subalgebra $\tilde{\mathfrak{h}}$ in the new basis $\tilde{e}_1 = -e_1 + e_2 - e_3$, $\tilde{e}_2 = \frac{1}{2}e_1 + \frac{1}{2}e_2 + e_3$, $\tilde{e}_3 = 2e_1 - e_2 + 2e_3$ is $\tilde{\mathfrak{h}} = \{-3\tilde{e_1} + 2\tilde{e_2} - 2\tilde{e_3}\}$. In Maple,

$> ChangeFrame(alg2) :$

$> h2 := DGzip(GC, [e1, e2, e3]);$

$$h2 := [-3e1 + 2e2 - 2e3]$$

Define the pair for this basis,

$> P2 := [alg2, h2];$

$$P2 := [alg2, [-3e1 + 2e2 - 2e3]]$$

and run the classifier on the new pair:

$> HomogeneousSpaceClassifier(P2);$

$$[[3, 2, 1]]$$

We can see a list of some of the properties that describe this Lie algebra-subalgebra pair by including the optional argument $properties = true$ to the program.

$> HomogeneousSpaceClassifier(P2, properties = true);$

```
Lie Algebra Dimension: 3
Algebra Type: Solvable
Indecomposable: true
Decomposition dimension: [3]
Derived Series Dimensions: [3, 2, 0]
Nilradical Dimension: 2
Nilradical Indecomposable: false
Derivations Dimension: 4
Dimensions of the Decomposition of the Radical of the Derivations: [4]
Derivations of Derivations Dimension: 4
Isotropy Type: F12
Isotropy Subalgebra and a Complementary Basis a Reductive Pair: true
Isotropy Subalgebra and a Complementary Basis a Symmetric Pair: true
Isotropy Subalgebra In Semisimple Part: NA
Subspace Type: NA
Orbit Dimension: 2
```

$$[[3, 2, 1]]$$

To view a brief outline of the steps the classifier is taking, one may use the Maple command $infolevel$, setting it to 2,

$> infolevel[HomogeneousSpaceClassifier] := 2 :$

$> HomogeneousSpaceClassifier(P2);$

```
Step 1: Classifying by Lie algebraic invariants.
Step 2: Classifying by invariants of ideals in derived series.
Step 3: Classifying by invariants of ideals in lower series.
Step 4: Classifying by orbit dimension.
Step 5: Classifying by properties of isotropy subalgebra.
```

$$[[3, 2, 1]]$$

To view a more detailed listing of the properties computed and the steps taken, set $infolevel$ to 3,

$> infolevel[HomogeneousSpaceClassifier] := 3 :$

$> HomogeneousSpaceClassifier(P2);$

```
Step 1: Classifying by Lie algebraic invariants.
      algebra type: "Solvable"
      derived series dimensions: [3, 2, 0]
      indecomposable: true
      dimension of nilradical: 2
      dimension of Lie algebra of derivations: 4
      dimension of Lie algebra of derivations of Lie algebra of derivations: 4
      dimensions of the decomposition of the radical of the Lie algebra of
    derivations: [4]
      dimensions of the decomposition of the Lie algebra: [3]
      indecomposable nilradical: false
Step 2: Classifying by invariants of ideals in derived series.
   Classifying by invariants of ideal (1) of derived series.
      algebra type: "Abelian"
      derived series dimensions: [2, 0]
      indecomposable: false
      dimension of nilradical: 2
      dimension of Lie algebra of derivations: 4
      dimension of Lie algebra of derivations of Lie algebra of derivations: 4
      dimensions of the decomposition of the radical of the Lie algebra of
    derivations: []
      dimensions of the decomposition of the Lie algebra: []
      indecomposable nilradical: false
```

```
    Classifying by invariants of ideal (1) quotient algebra.
      algebra type: "Abelian"
      derived series dimensions: [1, 0]
      indecomposable: true
      dimension of nilradical: 1
      dimension of Lie algebra of derivations: 1
      dimension of Lie algebra of derivations of Lie algebra of derivations: 1
      dimensions of the decomposition of the radical of the Lie algebra of
     derivations: []
      dimensions of the decomposition of the Lie algebra: []
      indecomposable nilradical: true
  Step 3: Classifying by invariants of ideals in lower series.
    Classifying by invariants of ideal (1) of lower series.
      algebra type: "Abelian"
      derived series dimensions: [2, 0]
      indecomposable: false
      dimension of nilradical: 2
      dimension of Lie algebra of derivations: 4
      dimension of Lie algebra of derivations of Lie algebra of derivations: 4
      dimensions of the decomposition of the radical of the Lie algebra of
     derivations: []
      dimensions of the decomposition of the Lie algebra: []
      indecomposable nilradical: false
    Classifying by invariants of ideal (1) quotient algebra.
      algebra type: "Abelian"
      derived series dimensions: [1, 0]
      indecomposable: true
      dimension of nilradical: 1
      dimension of Lie algebra of derivations: 1
      dimension of Lie algebra of derivations of Lie algebra of derivations: 1
      dimensions of the decomposition of the radical of the Lie algebra of
     derivations: []
      dimensions of the decomposition of the Lie algebra: []
      indecomposable nilradical: true
    Classifying by invariants of ideal (2) of lower series.
      algebra type: "Abelian"
      derived series dimensions: [2, 0]
      indecomposable: false
      dimension of nilradical: 2
      dimension of Lie algebra of derivations: 4
      dimension of Lie algebra of derivations of Lie algebra of derivations: 4
      dimensions of the decomposition of the radical of the Lie algebra of
     derivations: []
      dimensions of the decomposition of the Lie algebra: []
```

```
      indecomposable nilradical: false
   Classifying by invariants of ideal (2) quotient algebra.
      algebra type: "Abelian"
      derived series dimensions: [1, 0]
      indecomposable: true
      dimension of nilradical: 1
      dimension of Lie algebra of derivations: 1
      dimension of Lie algebra of derivations of Lie algebra of derivations: 1
      dimensions of the decomposition of the radical of the Lie algebra of
    derivations: []
      dimensions of the decomposition of the Lie algebra: []
      indecomposable nilradical: true
Step 4: Classifying by orbit dimension.
   Reducing list by orbit dimension: 2
Step 5: Classifying by properties of isotropy subalgebra.
      reductive Lie algebra-subalgebra pair: true
      symmetric Lie algebra-subalgebra pair: true
```

$$[[3, 2, 1]]$$

To view the above details in addition to a view of the classifier reducing the list of possible pairs, use $infolevel$ set to 4,

$> infolevel[HomogeneousSpaceClassifier] := 4 :$

$> HomogeneousSpaceClassifier(P2);$

```
Step 1: Classifying by Lie algebraic invariants.
   List to reduce: [[3, 2, 1], [3, 2, 2], [3, 2, 3], [3, 2, 4], [3, 2, 5], [3, 3, 1],
     [3, 3, 2], [3, 3, 3], [3, 3, 4], [3, 3, 5], [3, 3, 6], [3, 3, 7], [3, 3, 8], [3,
     3, 9]]
   Reducing list by...
      algebra type: "Solvable"
         [[3, 2, 1], [3, 2, 4], [3, 3, 1], [3, 3, 4], [3, 3, 5], [3, 3, 6], [3, 3,
    7]]
      derived series dimensions: [3, 2, 0]
         [[3, 2, 1], [3, 2, 4], [3, 3, 4], [3, 3, 5], [3, 3, 6], [3, 3, 7]]
      indecomposable: true
         [[3, 2, 1], [3, 2, 4], [3, 3, 4], [3, 3, 5], [3, 3, 6], [3, 3, 7]]
      dimension of nilradical: 2
         [[3, 2, 1], [3, 2, 4], [3, 3, 4], [3, 3, 5], [3, 3, 6], [3, 3, 7]]
      dimension of Lie algebra of derivations: 4
         [[3, 2, 1], [3, 2, 4], [3, 3, 4], [3, 3, 6], [3, 3, 7]]
      dimension of Lie algebra of derivations of Lie algebra of derivations: 4
```

```
      [[3, 2, 1], [3, 2, 4], [3, 3, 4], [3, 3, 7]]
    dimensions of the decomposition of the radical of the Lie algebra of
  derivations: [4]
      [[3, 2, 1], [3, 3, 7]]
    dimensions of the decomposition of the Lie algebra: [3]
      [[3, 2, 1], [3, 3, 7]]
    indecomposable nilradical: false
      [[3, 2, 1], [3, 3, 7]]
Step 2: Classifying by invariants of ideals in derived series.
  Classifying by invariants of ideal (1) of derived series.
  List to reduce: [[3, 2, 1], [3, 3, 7]]
  Reducing list by...
    algebra type: "Abelian"
      [[3, 2, 1], [3, 3, 7]]
    derived series dimensions: [2, 0]
      [[3, 2, 1], [3, 3, 7]]
    indecomposable: false
      [[3, 2, 1], [3, 3, 7]]
    dimension of nilradical: 2
      [[3, 2, 1], [3, 3, 7]]
    dimension of Lie algebra of derivations: 4
      [[3, 2, 1], [3, 3, 7]]
    dimension of Lie algebra of derivations of Lie algebra of derivations: 4
      [[3, 2, 1], [3, 3, 7]]
    dimensions of the decomposition of the radical of the Lie algebra of
  derivations: []
      [[3, 2, 1], [3, 3, 7]]
    dimensions of the decomposition of the Lie algebra: []
      [[3, 2, 1], [3, 3, 7]]
    indecomposable nilradical: false
      [[3, 2, 1], [3, 3, 7]]
  Classifying by invariants of ideal (1) quotient algebra.
  List to reduce: [[3, 2, 1], [3, 3, 7]]
  Reducing list by...
    algebra type: "Abelian"
      [[3, 2, 1], [3, 3, 7]]
    derived series dimensions: [1, 0]
      [[3, 2, 1], [3, 3, 7]]
    indecomposable: true
      [[3, 2, 1], [3, 3, 7]]
    dimension of nilradical: 1
      [[3, 2, 1], [3, 3, 7]]
    dimension of Lie algebra of derivations: 1
      [[3, 2, 1], [3, 3, 7]]
```

```
      dimension of Lie algebra of derivations of Lie algebra of derivations: 1
         [[3, 2, 1], [3, 3, 7]]
      dimensions of the decomposition of the radical of the Lie algebra of
    derivations: []
         [[3, 2, 1], [3, 3, 7]]
      dimensions of the decomposition of the Lie algebra: []
         [[3, 2, 1], [3, 3, 7]]
      indecomposable nilradical: true
         [[3, 2, 1], [3, 3, 7]]
Step 3: Classifying by invariants of ideals in lower series.
   Classifying by invariants of ideal (1) of lower series.
   List to reduce: [[3, 2, 1], [3, 3, 7]]
   Reducing list by...
      algebra type: "Abelian"
         [[3, 2, 1], [3, 3, 7]]
      derived series dimensions: [2, 0]
         [[3, 2, 1], [3, 3, 7]]
      indecomposable: false
         [[3, 2, 1], [3, 3, 7]]
      dimension of nilradical: 2
         [[3, 2, 1], [3, 3, 7]]
      dimension of Lie algebra of derivations: 4
         [[3, 2, 1], [3, 3, 7]]
      dimension of Lie algebra of derivations of Lie algebra of derivations: 4
         [[3, 2, 1], [3, 3, 7]]
      dimensions of the decomposition of the radical of the Lie algebra of
    derivations: []
         [[3, 2, 1], [3, 3, 7]]
      dimensions of the decomposition of the Lie algebra: []
         [[3, 2, 1], [3, 3, 7]]
      indecomposable nilradical: false
         [[3, 2, 1], [3, 3, 7]]
   Classifying by invariants of ideal (1) quotient algebra.
   List to reduce: [[3, 2, 1], [3, 3, 7]]
   Reducing list by...
      algebra type: "Abelian"
         [[3, 2, 1], [3, 3, 7]]
      derived series dimensions: [1, 0]
         [[3, 2, 1], [3, 3, 7]]
      indecomposable: true
         [[3, 2, 1], [3, 3, 7]]
      dimension of nilradical: 1
         [[3, 2, 1], [3, 3, 7]]
      dimension of Lie algebra of derivations: 1
```

```
      [[3, 2, 1], [3, 3, 7]]
   dimension of Lie algebra of derivations of Lie algebra of derivations: 1
      [[3, 2, 1], [3, 3, 7]]
   dimensions of the decomposition of the radical of the Lie algebra of
 derivations: []
      [[3, 2, 1], [3, 3, 7]]
   dimensions of the decomposition of the Lie algebra: []
      [[3, 2, 1], [3, 3, 7]]
   indecomposable nilradical: true
      [[3, 2, 1], [3, 3, 7]]
Classifying by invariants of ideal (2) of lower series.
List to reduce: [[3, 2, 1], [3, 3, 7]]
Reducing list by...
   algebra type: "Abelian"
      [[3, 2, 1], [3, 3, 7]]
   derived series dimensions: [2, 0]
      [[3, 2, 1], [3, 3, 7]]
   indecomposable: false
      [[3, 2, 1], [3, 3, 7]]
   dimension of nilradical: 2
      [[3, 2, 1], [3, 3, 7]]
   dimension of Lie algebra of derivations: 4
      [[3, 2, 1], [3, 3, 7]]
   dimension of Lie algebra of derivations of Lie algebra of derivations: 4
      [[3, 2, 1], [3, 3, 7]]
   dimensions of the decomposition of the radical of the Lie algebra of
 derivations: []
      [[3, 2, 1], [3, 3, 7]]
   dimensions of the decomposition of the Lie algebra: []
      [[3, 2, 1], [3, 3, 7]]
   indecomposable nilradical: false
      [[3, 2, 1], [3, 3, 7]]
Classifying by invariants of ideal (2) quotient algebra.
List to reduce: [[3, 2, 1], [3, 3, 7]]
Reducing list by...
   algebra type: "Abelian"
      [[3, 2, 1], [3, 3, 7]]
   derived series dimensions: [1, 0]
      [[3, 2, 1], [3, 3, 7]]
   indecomposable: true
      [[3, 2, 1], [3, 3, 7]]
   dimension of nilradical: 1
      [[3, 2, 1], [3, 3, 7]]
   dimension of Lie algebra of derivations: 1
```

```
          [[3, 2, 1], [3, 3, 7]]
       dimension of Lie algebra of derivations of Lie algebra of derivations: 1
          [[3, 2, 1], [3, 3, 7]]
       dimensions of the decomposition of the radical of the Lie algebra of
     derivations: []
          [[3, 2, 1], [3, 3, 7]]
       dimensions of the decomposition of the Lie algebra: []
          [[3, 2, 1], [3, 3, 7]]
       indecomposable nilradical: true
          [[3, 2, 1], [3, 3, 7]]
Step 4: Classifying by orbit dimension.
    Reducing list by orbit dimension: 2
       [[3, 2, 1]]
Step 5: Classifying by properties of isotropy subalgebra.
    List to reduce: [[3, 2, 1]]
    Reducing list by...
       reductive Lie algebra-subalgebra pair: true
       [[3, 2, 1]]
       symmetric Lie algebra-subalgebra pair: true
       [[3, 2, 1]]
       [[3, 2, 1]]
```

$$[[3, 2, 1]]$$

We can prove the results of the classifier by using the program *IsomorphismForLiePairs*.

$> ans := IsomorphismForLiePairs(P1, P2);$

$$ans := \left[\begin{bmatrix} 3 & -1 & -3 \\ -1 & 1 & 2 \\ 2 & -1 & -2 \end{bmatrix}\right]$$

The result is a matrix giving the equivalence of the Lie algebra-subalgebra pairs. Use the matrix to define a linear transformation $\psi : \mathfrak{g} \to \tilde{\mathfrak{g}}$,

$> \psi := LinearTransformation(alg1, alg2, ans[1]);$

$$\psi := e1 \to -3e1 + e2 - 2e3\ , e2 \to e1 - e2 + e3,\ e3 \to -3e1 + 2e2 - 2e3$$

This transformation gives a basis $\hat{e}_i = a^k \tilde{e}_k$ of $\tilde{\mathfrak{g}}$ ($alg2$ in Maple), such that the structure equations $[\hat{e}_i, \hat{e}_j] = C^k_{i,j} \hat{e}_k$ are those of $\mathfrak{g}$ ($alg1$ in Maple), where $[e_i, e_j] = C^k_{i,j} e_k$ for $e_i \in \mathfrak{g}$.

$> ChangeFrame(alg1):$

$> NB2 := ApplyLinearTransformation(psi, [e1, e2, e3]);$

$$NB2 := [3e1 - e2 + 2e3, e1 - e2 + e3, 3e1 - 2e2 + 2e3]$$

Initializing the basis, which is comprised of vectors in $\tilde{\mathfrak{g}}$ ($alg2$), shows the structure equations are indeed those of $\mathfrak{g}$ ($alg1$).

$> LD3 := LieAlgebraData(NB2, alg3); DGsetup(LD3)$

$$LD3 := [e1, e2] = 0,\ [e1, e3] = -e2,\ [e2, e3] = e1$$

Clearly in this example, ψ sends $\mathfrak{h}$ to $\tilde{\mathfrak{h}}$,

$> ChangeFrame(alg1):$

$> ApplyLinearTransformation(psi, P1[2]);$

$$[3e1 - 2e2 + 2e3]$$

Therefore the two pairs are equivalent.

4.4 Conclusion

It should be noted that the database has been classified successfully and uniquely against itself. To do this, the techniques described above for creating changes of basis on pairs were applied on each pair in the database, under randomly chosen matrices. Then the pair with its new structure equations were input to the classifier for testing. Thus the properties chosen classify fully the Lorentzian Lie algebra-subalgebra pairs of dimension three through seven. It is an interesting problem to try to uncover a minimum set of algebraic invariants for the classification of Lorentzian pairs. We hope to study this problem in the near future.

This concludes the discussion of the software developed for the classification of Lorentzian pairs. The program *HomogeneousSpaceClassifier* can be used to access the Lorentzian Lie algebra-subalgebra pairs of Chapter 3, assembled in a database file called *HomogeneousSpaceMainTable*. It is also the tool used to classify such pairs against the database. The program *IsomorphismForLiePairs* can be used in attempts to find explicit isomorphisms for pairs of interest. All database and code files can be found at Digital Commons at Utah State University for public download and use.

CHAPTER 5

SYMMETRIES OF SPACETIMES

The Lorentzian Lie algebra-subalgebra pairs of Chapter 3 will now be associated with four-dimensional simple G spacetimes. Vector field systems Γ defining the symmetries of the spacetimes will be constructed. Locally these are the infinitesimal isometries giving the isometry-isotropy algebra-subalgebra pair of a four-dimensional Lorentzian metric tensor g and are equivalent abstractly to the Lorentzian Lie algebra-subalgebra pairs of Chapter 3. Isomorphisms on each real Lie algebra Γ will be found that give bases in which the structure equations exhibit certain characteristics. First, we give a basis in which the subset of vector fields defining the isotropy subalgebra are placed as the last vector fields and in which the adjoint representation restricted to a complement of the isotropy is in standard form (see Table 5.1). Second, we give a basis in which the classification of the Lie algebra is manifest and easily identified in the literature (for example in Bowers [4], Fels [5], Rozum [10], Ŝnobl [11]).

Furthermore, for each Γ we will construct a basis $\mathfrak{G}$ of invariant symmetric rank-2 covariant tensors over the ring of Γ-invariant functions. These are tensors σ preserved under the Lie derivative by all vectors $X \in \Gamma$, that is, $\mathcal{L}_X \sigma = 0$. A general metric tensor field $g \in \mathrm{span}(\mathfrak{G})$ will then be constructed and normalized (or gauge fixed). The process of normalization will be described in Section 5.3 and utilizes the normalizer of Γ in the Lie algebra of vector fields $\mathfrak{X}(M)$ which fixes $\mathfrak{G}$ and provides freedom to possibly remove extraneous parameters and functions from the local coordinate presentation of g. Normalization will not be performed on metrics admitting three-dimensional and four-dimensional Lie algebras with trivial isotropy and will be a future project. The vector fields, preferred points, invariant quadratic forms, normalizers, and normalizations are stored in a database. These are the principal results of this dissertation and are presented in Appendix A.2.

The program *HomogeneousSpaceClassifier* of Chapter 4 has been adapted to create a program called *SpaceTimeSymmetryClassifier*. This simply takes as input a basis of Killing vectors and a point at which to compute the isotropy subalgebra. The Killing vectors in a neighborhood of a point define an abstract Lorentzian Lie algebra-subalgebra pair that

Table 5.1: Adjoint representations of isotropy subalgebras and the isotropy type of each, abstractly defining subalgebras of $\mathfrak{so}(3,1)$.

$\mathbf{G_r}$ **on** $\mathbf{V_k = G/H}$	**Adjoint Representation**	**Isotropy Type**
G_3 on V_2	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$F12$, $F13$, resp.
G_4 on V_3	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$	$F12$, $F13$, $F14$, resp.
G_5 on V_4	$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	$F12$, $F13$, $F14$, resp.
G_6 on V_3	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$	$F3$
G_6 on V_3	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	$F4$
G_6 on V_4	$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$	$F9$
G_6 on V_4	$\begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$	$F10$
G_7 on V_4	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	F3
G_7 on V_4	$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	F4
G_7 on V_4	$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$	$F6$

can be classified by *HomogeneousSpaceClassifier* against the database of such pairs. In this way we can classify the symmetries of spacetimes that are simple G spaces, for G the Lie group of isometries, against the classification of such spacetimes created in this chapter.

An example will be given that demonstrates the process of constructing vector fields, their invariant quadratic forms, and normalizing the invariant quadratic forms.

5.1 Vector field systems for Lorentzian pairs

The first approach in associating vector field systems to the Lorentzian pairs of Chapter 3 is to construct them directly from first principles. Let G be an r-dimensional Lie group, H a closed subgroup, and G/H a k-dimensional homogeneous space with $(\mathfrak{g}, \mathfrak{h})$ the respective Lie algebras. We begin by constructing a basis R of right-invariant vector fields on G. Note that we won't have need to explicitly compute G as we employ Cartan's formula (2.1) to first compute right-invariant one-forms on G, from which is computed the right-invariant vector fields (see Section 5.1.1). Then these vector fields are pushed forward to the homogeneous space G/H under the projection map $\pi : G \to G/H$. Note $\Gamma = \{X_i = \pi_*(R_i) \mid R_i \in R\}$ will have identical structure equations to $\mathfrak{g}$. To compute π, the left-invariant vector fields L are needed. The pushforward will require a smooth local cross-section $\iota : G/H \to G$ such that $\pi \circ \iota = \mathbb{1}_{G/H}$. An example giving all details is worked completely in Section 5.1.1.

The above approach works best when dealing with solvable Lie algebras as in those cases the equations are easily solved (see the first example below). Thus a second approach could be inductive in nature. If working with a semisimple or generic Lie algebra, first find a solvable subalgebra of highest possible dimension containing $\mathfrak{h}$. Applying the first approach to the solvable subalgebra, one easily constructs vector fields defining it. Then by defining an arbitrary vector field (or vector fields), one may enforce the conditions imposed by the structure equations of $\mathfrak{g}$ and the action of the isotropy to give PDE which as a practical matter can be solved to give explicitly the remaining vector field (or vector fields). However, this approach of finding a solvable subalgebra was not necessary as the majority of semisimple or generic Lorentzian pairs are decomposable and we mention it here only for informational purposes. For instance, for the semisimple Lorentzian pairs $[6, 3, 1]$ and $[6, 3, 4]$, which are $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ and $\mathfrak{so}(2, 1) \oplus \mathfrak{so}(2, 1)$ respectively, it was more convenient to use vector fields for $\mathfrak{so}(3)$ (or respectively for $\mathfrak{so}(2, 1)$) from the case of G_3 on V_3. Then finding three vector fields which commute with $\mathfrak{so}(3)$ (or $\mathfrak{so}(2, 1)$) conveniently gives another set of vector fields also defining $\mathfrak{so}(3)$ (or $\mathfrak{so}(2, 1)$) and at each point the isotropy of the full set of vector fields defining $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$ is $\mathfrak{so}(3)$ (or $\mathfrak{so}(2, 1)$) as it should be. For indecomposable semisimple or generic cases the following method was useful.

A third approach or method is to search the literature for vector field systems giving the pairs of our classification of Chapter 3. The book *Einstein Spaces* by Petrov [2] is one such resource. In this book, Petrov claims to have given a complete classification of spacetimes with symmetry according to local group action. By computing the abstract Lorentzian Lie algebra-subalgebra pairs from Petrov's Killing vectors and using the classifier developed in Chapter 4, we can associate vector fields to the pairs rather effortlessly. However, as will be discussed in more detail in the next chapter, Petrov's classification contains many gaps. Also, numerous vector field systems therein include superfluous parameters rendering those vector fields less convenient and undesirable. Therefore, for many of the vector field systems in the database created for this chapter, the first approach above was more appropriate.

As mentioned, for the abstract Lorentzian pairs given by the vector field systems, isomorphisms will be provided that do one of two things. First, an isomorphism that displays the properties of the isotropy and its action on a complement. If the Lorentzian pair is reductive, then in this new basis the adjoint representation of basis vectors defining the isotropy $\mathfrak{h}$ is computed, restricted to a reductive complement $\mathfrak{m}$. This gives a list of matrices in standard form that can be checked in Table 5.1. If the pair is non-reductive, we simply extract the upper left 4 by 4 sub-matrix of each representation matrix in the adjoint representation of the isotropy. In either case, the vectors defining the complement will be first in the basis, and the isotropy vectors last. Second, we provide an isomorphism that simply puts the vector fields in a basis whose structure equations are classified in the literature and for which a researcher can easily identify.

5.1.1 Example of homogeneous space

We will construct vector fields for the Lorentzian pair $[5, 4, 8]$ using the first approach. First we call the structure equations for the pair $[5, 4, 8]$:

$$> LD, P := HomogeneousSpaceClassifier(retrieve = [[5, 4, 8], alg]);$$

$$LD, P := [e1, e2] = 0,\ [e1, e3] = 0,\ [e1, e4] = -e1,\ [e1, e5] = 0,\ [e2, e3] = e1,$$

$$[e2, e4] = 0,\ [e2, e5] = e2,\ [e3, e4] = -e3,\ [e3, e5] = -e3,\ [e4, e5] = 0, [alg, [e5]]$$

Note the isotropy subalgebra is given by the second element of the list P, namely e_5. Next we compute the right invariant vector fields on G. To do this, recall Lie's Third Theorem, which guarantees, given a Lie algebra $\mathfrak{g}$ with structure constants C, there exists locally a Lie algebra of r pointwise independent vector fields with structure constants C on an r-dimensional manifold M (see Flanders [21]). The routine *LiesThirdTheorem*, in the subpackage GroupActions of the DifferentialGeometry package in Maple, applies Lie's Third Theorem and generates the vector fields we need. As of this writing, *LiesThirdTheorem* is only available for the special case of solvable Lie algebras. Note that in our example $M = G$. We will briefly demonstrate the basic process used by *LiesThirdTheorem* in constructing the right invariant vector fields R. (For an alternative approach see Fels [22].)

To begin we first determine the right invariant 1-forms θ^i on G. These are of course dual to the right invariant vector fields R_i on G and thus satisfy $\theta^i(R_j) = \delta^i_j$. Therefore let $\{\theta^i\}$ be the dual basis to the basis $\{e_i\}$ of $\mathfrak{g}$ and let z^i denote the coordinates on G. We then consider the θ^i to be dual to the right invariant vector fields on G, namely we consider $\theta^i = F^i_j(z^1, \ldots, z^5)dz^j$ where the F^i_j are unknown functions for which we must solve. By Cartan's formula (see equation (2.1)) and the structure equations given by LD above, we have

$$
\begin{aligned}
d\theta^1 &= \theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3, & \qquad d\theta^4 &= 0,\\
d\theta^2 &= -\theta^2 \wedge \theta^5, & \qquad d\theta^5 &= 0.\\
d\theta^3 &= \theta^3 \wedge \theta^4 + \theta^3 \wedge \theta^5, & &
\end{aligned}
$$

From these equations we see we can set $\theta^5 = dz^5$ and $\theta^4 = dz^4$. Then set $\theta^3 = dz^3 + A(z^1, \ldots, z^5)dz^4 + B(z^1, \ldots, z^5)dz^5$ for some functions A and B. Taking the exterior derivative and demanding the equation $d\theta^3 = \theta^3 \wedge \theta^4 + \theta^3 \wedge \theta^5$ be satisfied gives the following system of PDE:

$$
\frac{\partial}{\partial z^1} A\left(z^1, \ldots, z^5\right) = 0,
$$

$$
\frac{\partial}{\partial z^2} A\left(z^1, \ldots, z^5\right) = 0,
$$

$$
\frac{\partial}{\partial z^3} A\left(z^1, \ldots, z^5\right) - 1 = 0,
$$

$$\frac{\partial}{\partial z^1} B\left(z^1, \ldots, z^5\right) = 0,$$

$$\frac{\partial}{\partial z^2} B\left(z^1, \ldots, z^5\right) = 0,$$

$$\frac{\partial}{\partial z^3} B\left(z^1, \ldots, z^5\right) - 1 = 0,$$

and

$$-\frac{\partial}{\partial z^5} A\left(z^1, \ldots, z^5\right) + \frac{\partial}{\partial z^4} B\left(z^1, \ldots, z^5\right) = -B\left(z^1, \ldots, z^5\right) + A\left(z^1, \ldots, z^5\right).$$

Solving the system we see $A\left(z^1, \ldots, z^5\right) = B\left(z^1, \ldots, z^5\right) = z^3$ by letting all resulting arbitrary functions be zero. Therefore $\theta^3 = z^4 dz^3 + z^5 dz^3 + dz^3$. Similarly, since $d\theta^2 = -\theta^2 \wedge \theta^5 = -\theta^2 \wedge dz^5$, we choose to set $\theta^2 = dz^2 + C(z^1, \ldots, z^5)dz^5$ for some function C. Thus the exterior derivative gives

$$d\theta^2 = \frac{\partial}{\partial z^i} C(z^1, \ldots, z^5) dz^i \wedge dz^5 \equiv -\theta^2 \wedge dz^5.$$

This results in an easily solved system of PDE with particular solution $C(z^1, \ldots, z^5) = -z^2$. Thus we have $\theta^2 = dz^2 - z^2 dz^5$. Now letting

$$\theta^1 = dz^1 + D_1(z^1, \ldots, z^5)dz^2 + D_2(z^1, \ldots, z^5)dz^3 + D_3(z^1, \ldots, z^5)dz^4,$$

then taking the exterior derivative and demanding $d\theta^1 = \theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3$, we arrive at the particular solution (again by letting the resulting arbitrary functions be zero) $D_1(z^1, \ldots, z^5) = z^3 z^5 + z^3$, $D_2(z^1, \ldots, z^5) = z^2 z^5$, and $D_3(z^1, \ldots, z^5) = z^2 z^3 z^5 + z^1$. Thus

$$\theta^1 = dz^1 + (z^3 z^5 + z^3)dz^2 + z^2 z^5 dz^3 + (z^2 z^3 z^5 + z^1)dz^4.$$

Then a solution is the following:

$$\theta^1 = dz^1 + (z^3 z^5 + z^3)dz^2 + z^2 z^5 dz^3 + (z^2 z^3 z^5 + z^1)dz^4,$$

$$\theta^2 = dz^2 - z^2 dz^5,$$

$$\theta^3 = z^4 dz^3 + z^5 dz^3 + dz^3,$$

$$\theta^4 = dz^4,$$
$$\theta^5 = dz^5.$$

If we let $R_i = f_i^j(z^1, \ldots, z^5)\frac{\partial}{\partial z^j}$, we can solve the system $\theta^i(R_j) = \delta^i_j$ for the f_i^j easily and arrive at the following right invariant vector fields:

$$R_1 = \partial_{z^1},$$
$$R_2 = (-z^3\, z^5 - z^3)\,\partial_{z^1} + \partial_{z^2},$$
$$R_3 = -z^2\, z^5\,\partial_{z^1} + \partial_{z^3},$$
$$R_4 = -z^1\,\partial_{z^1} - z^3\,\partial_{z^3} + \partial_{z^4},$$
$$R_5 = -z^2\, z^3\,\partial_{z^1} + z^2\,\partial_{z^2} - z^3\,\partial_{z^3} + \partial_{z^5}$$

This is precisely the result of a call to the routine *LiesThirdTheorem*. To see this, first we set up a manifold to represent the Lie group G, choosing to use coordinates z^i, then simply apply *LiesThirdTheorem*, noting that we supply the routine with the name *alg* of our initialized Lie algebra and the name G for the coordinates:

$> DGsetup([z1, z2, z3, z4, z5], G):$

$> R := LiesThirdTheorem(alg, G);$

$$R := [\,\partial_{z1}, (-z3\ z5 - z3)\,\partial_{z1} + \partial_{z2}, -z2\ z5\,\partial_{z1} + \partial_{z3}, -z1\,\partial_{z1} - z3\,\partial_{z3} + \partial_{z4}, - z2\ z3\,\partial_{z1} + z2\,\partial_{z2} - z3\,\partial_{z3} + \partial_{z5}]$$

Observe that the structure equations for this Lie algebra of vector fields are precisely that of $[5, 4, 8]$:

$> LieAlgebraData(R);$

$$[e1, e2] = 0,\ [e1, e3] = 0,\ [e1, e4] = -e1,\ [e1, e5] = 0,\ [e2, e3] = e1,\ [e2, e4] = 0, [e2, e5] = e2,\ [e3, e4] = -e3,\ [e3, e5] = -e3,\ [e4, e5] = 0$$

We next need the left invariant vector fields L. These vector fields commute with R and therefore $[X, Y] = \mathcal{L}_X Y = 0$ for all $X \in L$ and $Y \in R$. The invaluable routine *InvariantGeometricObjectFields* will compute the Lie derivative for all $Y \in R$ with respect to an arbitrary vector field X with unknown functions of the coordinates z^i as coefficients. It then solves the resulting PDE and returns a basis for the invariant vector fields over the ring of invariant functions.

$> L0 := InvariantGeometricObjectFields(R,\ DGinfo(G,\ \text{“}FrameBaseVectors\text{”}),$
$output = \text{“}list\text{”});$

$$L0 := [-z2\ z3\ \partial_{z1} + \partial_{z5},\ \partial_{z4}, -\mathrm{e}^{-z5-z4}\, z2\ (z5+1)\ \partial_{z1} + \mathrm{e}^{-z5-z4}\ \partial_{z3},$$
$$-\,\mathrm{e}^{z5}\, z3\ z5\ \partial_{z1} + \mathrm{e}^{z5}\ \partial_{z2}, \mathrm{e}^{-z4}\ \partial_{z1}]$$

These are our left invariant vector fields, linearly independent over $\mathbb{R}$. However, the structure equations should be that of R, differing by a minus sign, but the solution from *InvariantGeometricObjectFields* was placed in no particular order. The following change of basis, amounting to a simple rearrangement of the vector fields in $L0$, gives the correct structure equations one would expect of left invariant vector fields:

$> L := evalDG([L0[5],\ -L0[4],\ -L0[3],\ L0[2],\ -L0[1]]);$

$$L := [\mathrm{e}^{-z4}\ \partial_{z1}, \mathrm{e}^{z5}\, z3\ z5\ \partial_{z1} - \mathrm{e}^{z5}\ \partial_{z2}, \mathrm{e}^{-z5-z4}\, z2\ (z5+1)\ \partial_{z1} - \mathrm{e}^{-z5-z4}\ \partial_{z3},\ \partial_{z4},$$
$$-\,z2\ z3\ \partial_{z1} + \partial_{z5}]$$

Now recall that the projection map $\pi : G \to G/H$ is given by $\pi(a) = aH$, that is, it sends the group element a to the coset aH. Thus, for any $h \in H$ we have

$$\pi(ah) = (ah)H = a(hH) = aH = \pi(a).$$

Therefore π is invariant under the right action of H on G. The infinitesimal generator of this right action is the left invariant vector field corresponding to the isotropy subalgebra, e_5, and is thus $Z = z^2\, z^3\, \partial_{z^1} - \partial_{z^5}$, the negative of the fifth vector field in the basis L.

$> Z := [L[5]];$

$$Z := [z2\, z3\, \partial_{z1} - \partial_{z5}]$$

Therefore, if

$$\pi(z^1, \ldots, z^5) = (F_1(z^1, \ldots, z^5), \ldots, F_5(z^1, \ldots, z^5)),$$

then it follows that each $F_1(z^1, \ldots, z^5), \ldots, F_5(z^1, \ldots, z^5)$ is an invariant function of Z, meaning $\mathcal{L}_Z(F_i) = Z(F_i) = 0$. In Maple, the command *LieDerivative* will generate the necessary PDE to solve, using an arbitrary function F.

$> PDE := LieDerivative(Z, F(z1, z2, z3, z4, z5));$

$$PDE := [z2\, z3\, \frac{\partial}{\partial z1} F\,(z1\,, z2\,, z3\,, z4\,, z5) - \frac{\partial}{\partial z5} F\,(z1\,, z2\,, z3\,, z4\,, z5)]$$

The built-in Maple command *pdsolve* solves the PDE:

$> sol := pdsolve(PDE);$

$$sol := \left\{F\,(z1\,, z2\,, z3\,, z4\,, z5) = \_F1\left(z2\,, z3\,, z4\,, \frac{z2\; z3\; z5 + z1}{z2\; z3}\right)\right\}$$

The quotient G/H, under the one-dimensional subgroup H infinitesimally generated by Z, is a four-dimensional manifold. We setup the manifold $M = G/H$ using coordinates x^i and define π:

$> DGsetup([x1, x2, x3, x4], M) :$

$> pi := Transformation(G, M, [x1 = z2, x2 = z3, x3 = z4, x4 = (z2 * z3 * z5 + z1)]);$

$$\pi := x1 = z2,\ x2 = z3,\ x3 = z4,\ x4 = z2z3z5 + z1$$

To use π to pushforward the right invariant vector fields R on G to G/H, we need a local cross-section $\iota : G/H \to G$. We find ι easily with the DifferentialGeometry command *InverseTransformation*, returning a local inverse:

$> iota := InverseTransformation(pi);$

$$\iota := z1 = -\_C1\; x1\; x2 + x4\,,\ z2 = x1\,,\ z3 = x2\,,\ z4 = x3\,,\ z5 = \_C1$$

noting $\_C1$ is an arbitrary constant. We check the inverse relationship of π and ι:

$> ComposeTransformations(pi, iota);$

$$x1 = x1,\ x2 = x2,\ x3 = x3,\ x4 = x4$$

Now pushforward the right invariant vector fields R to the homogeneous space G/H:

$> Gamma := Pushforward(pi, iota, R);$

$$\Gamma := [\,\partial_{x4},\ \partial_{x1} - x2\,\partial_{x4},\ \partial_{x2}, -x2\,\partial_{x2} + \partial_{x3} - x4\,\partial_{x4}, x1\,\partial_{x1} - x2\,\partial_{x2}]$$

The vector fields Γ are defined on a four-dimensional manifold with one-dimensional isotropy of type $F13$, giving precisely the Lorentzian Lie algebra-subalgebra Pair $[5, 4, 8]$ of Chapter 3. The identity isomorphism has this Lie algebra into the readily verified form of $\mathfrak{s}_{5,44}$, found in Snobl [11].

We also want a basis that manifests the nature and action of the isotropy. To achieve this, we place the basis vectors spanning the isotropy subalgebra in the last position (in this example this is already the case at the origin) and in which the adjoint representation restricted to a (reductive) complement of the isotropy subalgebra is in standard form (see Table 5.1). The following change of basis does just that:

$> LDL := LieAlgebraData([\Gamma[1],\ \Gamma[4],\ \Gamma[2] - \Gamma[3],\ \Gamma[2] + \Gamma[3],\ -\Gamma[5]], algL);$

$> DGsetup(LDL):$

$$LDL := [e1, e2] = -e1,\ [e1, e3] = 0,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e2, e3] = \frac{1}{2}e3 - \frac{1}{2}e4,$$
$$[e2, e4] = -\frac{1}{2}e3 + \frac{1}{2}e4,\ [e2, e5] = 0,\ [e3, e4] = 2e1,\ [e3, e5] = -e4,$$
$$[e4, e5] = -e3$$

This change of basis was found by solving for a matrix P such that $AP - PB = 0$, where A is the adjoint matrix of the isotropy restricted to a reductive complement and B is the standard form matrix in Table 5.1. While the structure equations themselves aren't necessarily that telling in this basis, we know the isotropy is given by the fifth vector field and we can compute the restricted adjoint representation and see immediately the isotropy type is $F13$ (refer to Table 5.1):

$> rep := Adjoint(e5, [e1, e2, e3, e4])$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$> IsotropyType([rep]);$

$F13$

5.2 Invariant quadratic forms

In the previous section a vector field system Γ was constructed which gives the Lorentzian Lie algebra-subalgebra pair $[5, 4, 8]$. This was done by dropping the right-invariant vector fields R on G to G/H under the natural projection map. In this next step we construct a basis $\mathfrak{G} \subset S^2T^*(M)$ of Γ-invariant symmetric rank-2 covariant tensors over the ring $\mathcal{I}(M)$ of Γ-invariant functions. These are tensors $\sigma_i \in \mathfrak{G}$ such that $\mathcal{L}_X\sigma_i = 0$ for all vectors $X \in \Gamma$. We do this by the following method.

Begin by constructing a dual basis of 1-forms ω^i for the left-invariant vector fields L on G. Since $\mathcal{L}_{R_j}\omega^i = 0$ for all right-invariant vector fields R_j, the ω^i are the invariant 1-forms for R. Therefore from these ω^i we construct the general R-invariant quadratic form $\gamma = \gamma_{ij}\omega^i \otimes \omega^j$ on G. We will pullback γ to G/H under the local cross-section $\iota : G/H \to G$ and impose ad(H)-invariance. This is done by setting $\mathcal{L}_Z\gamma = 0$, where Z denotes a left-invariant vector field which is an infinitesimal generator for the right action of the isotropy subgroup H on G. We will see in the following discussion that $\mathcal{L}_Z\gamma = 0$ gives *algebraic* conditions on the γ_{ij}. However, when these conditions are met and we pullback by ι, we will have the ad(H)-invariant quadratic forms on G/H.

Recall that the Lie algebra $\mathfrak{g}$ of G is written $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ where $\mathfrak{m}$ is a reductive complement to $\mathfrak{h}$, the Lie algebra of the isotropy subgroup H. Let $\{m_a\}$ and $\{h_\alpha\}$ be the basis elements of $\mathfrak{g}$ corresponding to $\mathfrak{m}$ and $\mathfrak{h}$ respectively, with $a = 1 \ldots \dim(\mathfrak{m})$ and $\alpha = 1 \ldots \dim(\mathfrak{h})$. Let the left-invariant vector fields L_a correspond to the abstract elements $\{m_a\}$ and L_α to those of $\{h_\alpha\}$. Note that $Z \in \{L_\alpha\}$. Let $\{\omega^i\}$ be the basis of 1-forms dual to L as in the previous paragraph. Then observe the Lie derivative of ω^i along the vector field L_α, evaluated at an arbitrary left-invariant vector field L_j:

$$\begin{aligned}
(\mathcal{L}_{L_\alpha}\omega^i)(L_j) &= L_\alpha(\omega^i(L_j)) - \omega^i([L_\alpha, L_j]) \\
&= L_\alpha(\delta^i_j) - \omega^i(-C^k_{\alpha j}L_k) \\
&= 0 + C^k_{\alpha j}\omega^i(L_k) \\
&= C^k_{\alpha j}\delta^i_k \\
&= C^i_{\alpha j}
\end{aligned}$$

Note the constants $C^i_{\alpha a}$ are the entries of the adjoint matrix given by h_α or $-L_\alpha$. Then observe the Lie derivative of γ:

$$\begin{aligned}
\mathcal{L}_{L_\alpha}\gamma &= \mathcal{L}_{L_\alpha}(\gamma_{ij}\omega^i \otimes \omega^j) \\
&= \mathcal{L}_{L_\alpha}(\gamma_{ij}\omega^i) \otimes \omega^j + \gamma_{ij}\omega^i \otimes \mathcal{L}_{L_\alpha}\omega^j \\
&= \gamma_{ij}\mathcal{L}_{L_\alpha}\omega^i \otimes \omega^j + \gamma_{ij}\omega^i \otimes \mathcal{L}_{L_\alpha}\omega^j \\
&= \gamma_{ij}C^i_{\alpha k}\omega^k \otimes \omega^j + \gamma_{ij}\omega^i \otimes C^j_{\alpha l}\omega^l \\
&= \gamma_{ij}C^i_{\alpha k}(\omega^k \otimes \omega^j) + \gamma_{ij}C^j_{\alpha l}(\omega^i \otimes \omega^l) \\
&= \gamma_{il}C^i_{\alpha k}(\omega^k \otimes \omega^l) + \gamma_{kj}C^j_{\alpha l}(\omega^k \otimes \omega^l) \\
&= (\gamma_{pl}C^p_{\alpha k} + \gamma_{kq}C^q_{\alpha l})\omega^k \otimes \omega^l,
\end{aligned}$$

noting the change of dummy indices. Evaluating at arbitrary left-invariant vector fields L_i and L_j and setting equal to zero we see

$$\begin{aligned}
(\gamma_{pl}C^p_{\alpha k} + \gamma_{kq}C^q_{\alpha l})\omega^k \otimes \omega^l(L_i, L_j) &= (\gamma_{pl}C^p_{\alpha k} + \gamma_{kq}C^q_{\alpha l})\omega^k(L_i)\omega^l(L_j) \\
&= (\gamma_{pl}C^p_{\alpha k} + \gamma_{kq}C^q_{\alpha l})\delta^k_i\delta^l_j \\
&= (\gamma_{pj}C^p_{\alpha i} + \gamma_{iq}C^q_{\alpha j})
\end{aligned}$$

$$= 0.$$

Then this set of conditions may be written in matrix notation:

$$C_\alpha^T \gamma + \gamma C_\alpha = 0. \tag{5.1}$$

Again note that C_α is the adjoint matrix of $-L_\alpha$.

Lastly, since $\pi^*(g)(L_i, L_j) = \gamma(\pi_*(L_i), \pi_*(L_j))$, if $L_i = L_\alpha$ or $L_j = L_\alpha$, then $\pi^*(g)(L_i, L_j) = 0$ and therefore basis elements $\omega^i \otimes \omega^j$ with a component of ω^α must drop to zero on G/H under ι. Indeed, from the general theory discussed in this section, we could have defined $\gamma = \gamma_{ab}\omega^a \otimes \omega^b$, for $a, b = 1 \ldots \dim(\mathfrak{m})$, from the outset.

We will now follow the above procedures on the example began in Section 5.1 and find a basis of Γ-invariant quadratic forms.

5.2.1 Example of homogeneous space continued

Recall the vector fields Γ from where we left off in the previous section:

$$X_1 = \partial_{x^4}, \qquad X_4 = -x^2\,\partial_{x^2} + \partial_{x^3} - x^4\,\partial_{x^4},$$
$$X_2 = \partial_{x^1} - x^2\,\partial_{x^4}, \qquad X_5 = x^1\,\partial_{x^1} - x^2\,\partial_{x^2}.$$
$$X_3 = \partial_{x^2},$$

Also recall the left-invariant vector fields L on G,

$$L_1 = \mathrm{e}^{-z^4}\,\partial_{z1}, \qquad L_4 = \partial_{z^4},$$
$$L_2 = \mathrm{e}^{z^5}\,z^3\,z^5\,\partial_{z^1} - \mathrm{e}^{z^5}\,\partial_{z^2}, \qquad L_5 = -z^2\,z^3\,\partial_{z^1} + \partial_{z^5},$$
$$L_3 = \mathrm{e}^{-z^5-z^4}\,z^2\,\left(z^5+1\right)\partial_{z^1} - \mathrm{e}^{-z^5-z^4}\,\partial_{z^3},$$

and the right-invariant vector fields R on G,

$$R_1 = \partial_{z^1}, \qquad R_4 = -z^1\,\partial_{z^1} - z^3\,\partial_{z^3} + \partial_{z^4},$$
$$R_2 = (-z^3\,z^5 - z^3)\,\partial_{z^1} + \partial_{z^2}, \qquad R_5 = -z^2\,z^3\,\partial_{z^1} + z^2\,\partial_{z^2} - z^3\,\partial_{z^3} + \partial_{z^5}.$$

$$R_3 = -z^5\, z^2\, \partial_{z^1} + \partial_{z^3},$$

We compute the dual basis ω^i to L in Maple by the following command, noting that we've simplified the result using *simplify*:

$> DB := simplify(DualBasis(L), symbolic)$

$$[\mathrm{e}^{z4}dz1 + z3z5\mathrm{e}^{z4}dz2 + z2(z5+1)\mathrm{e}^{z4}dz3 + z3z5\mathrm{e}^{z4}dz2, -\mathrm{e}^{-z5}dz2,$$
$$-\,\mathrm{e}^{z5+z4}dz3, dz4, dz5]$$

The 1-forms are thus

$$\begin{aligned}
\omega^1 &= \mathrm{e}^{z^4}dz^1 + z^3 z^5 \mathrm{e}^{z^4}dz^2 + z^2(z^5+1)\mathrm{e}^{z^4}dz^3 + z^3 z^5 \mathrm{e}^{z^4}dz^2,\\
\omega^2 &= -\mathrm{e}^{-z^5}dz^2,\\
\omega^3 &= -\mathrm{e}^{z^5+z^4}dz^3,\\
\omega^4 &= dz^4,\\
\omega^5 &= dz^5.
\end{aligned}$$

Observe the adjoint representation of $Z = -L_5$, the infinitesimal generator of the right action of H on G (as discussed in Section 5.1):

$$ad(Z) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In the discussion above in equation (5.1), note that $C_\alpha = ad(Z)$. Then define γ_{ij} by the following command:

$> gamma\_ij := Matrix(4, 4, shape = symmetric, symbol = 's');$

$$gamma\_ij := \begin{bmatrix} s_{1,1} & s_{1,2} & s_{1,3} & s_{1,4} & s_{1,5} \\ s_{1,2} & s_{2,2} & s_{2,3} & s_{2,4} & s_{2,5} \\ s_{1,3} & s_{2,3} & s_{3,3} & s_{3,4} & s_{3,5} \\ s_{1,4} & s_{2,4} & s_{3,4} & s_{4,4} & s_{4,5} \\ s_{1,5} & s_{2,5} & s_{3,5} & s_{4,5} & s_{5,5} \end{bmatrix}$$

Then using equation (5.1), we arrive at

$> Transpose(AdZ).gamma\_ij + gamma\_ij.AdZ;$

$$\begin{bmatrix} 0 & s_{1,2} & -s_{1,3} & 0 & 0 \\ s_{1,2} & 2\,s_{2,2} & 0 & s_{2,4} & s_{2,5} \\ -s_{1,3} & 0 & -2\,s_{3,3} & -s_{3,4} & -s_{3,5} \\ 0 & s_{2,4} & -s_{3,4} & 0 & 0 \\ 0 & s_{2,5} & -s_{3,5} & 0 & 0 \end{bmatrix}$$

Setting this matrix equal to the zero matrix we see immediately that

$$\{s_{1,2} = 0, s_{1,3} = 0, s_{2,2} = 0, s_{2,4} = 0, s_{2,5} = 0, s_{3,3} = 0, s_{3,4} = 0, s_{3,5} = 0\}$$

Forcing this solution into γ_{ij} we have

$$(\gamma_{ij}) = \begin{bmatrix} s_{1,1} & 0 & 0 & s_{1,4} & s_{1,5} \\ 0 & 0 & s_{2,3} & 0 & 0 \\ 0 & s_{2,3} & 0 & 0 & 0 \\ s_{1,4} & 0 & 0 & s_{4,4} & s_{4,5} \\ s_{1,5} & 0 & 0 & s_{4,5} & s_{5,5} \end{bmatrix} \tag{5.2}$$

giving the most general R-invariant metric on G that is invariant under $Z = L_5$ as $\gamma = \gamma_{ij}\omega^i \otimes \omega^j$. We are now able to drop γ to a Γ-invariant metric g on G/H by pullback under the local cross-section ι from Section 5.1. In Maple we can form γ as follows, though we suppress the output here as its length outweighs its usefulness:

$> gamma\_1 := add(add(evalDG(gamma\_ij[i,j] * DB[i] \&tensor\ DB[j]),$
$i = 1..nops(DB)), j = 1..nops(DB));$

Recall DB was the basis of dual 1-forms ω^i to the left-invariant vector fields. We are ready to pullback γ:

$> g := Pullback(iota, gamma\_1);$

$$g := s_{2,3}\mathrm{e}^{x3}\, dx^1\, dx^2 + s_{2,3}\mathrm{e}^{x3}\, dx^2\, dx^1 + s_{1,1}\mathrm{e}^{2\,x3}\, x1^2\, dx^2\, dx^2 + s_{1,4}\mathrm{e}^{x3}\, x1\, dx^2\, dx^3$$
$$+ s_{1,1}\mathrm{e}^{2\,x3}\, x1\, dx^2\, dx^4 + s_{1,4}\mathrm{e}^{x3}\, x1\, dx^3\, dx^2 + s_{4,4}\, dx^3\, dx^3 + s_{1,4}\mathrm{e}^{x3}\, dx^3\, dx^4$$
$$+ s_{1,1}\mathrm{e}^{2\,x3}\, x1\, dx^4\, dx^2 + s_{1,4}\mathrm{e}^{x3}\, dx^4\, dx^3 + s_{1,1}\mathrm{e}^{2\,x3}\, dx^4\, dx^4$$

This is the most general Γ-invariant metric on G/H and one may easily check in Maple that $\mathcal{L}_X g = 0$ for all $X \in \Gamma$. Also, the isometry dimension of g is easily verified to be five by running the command *KillingVectors(g)*. We may write this metric in symmetric tensor product notation as follows, where $dx^i \odot dx^j = dx^i \otimes dx^j + dx^j \otimes dx^i$:

$$g = s_{2,3}\mathrm{e}^{x^3}\, dx^1 \odot dx^2 + s_{1,1}{x^1}^2\mathrm{e}^{2\,x^3}\, dx^2 \odot dx^2 + s_{1,4}x^1\mathrm{e}^{x^3}\, dx^2 \odot dx^3$$
$$+ s_{1,1}x^1\mathrm{e}^{2\,x^3}\, dx^2 \odot dx^4 + s_{4,4}\, dx^3 \odot dx^3 + s_{1,4}\mathrm{e}^{x^3}\, dx^3 \odot dx^4 + s_{1,1}\mathrm{e}^{2\,x3}\, dx^4 \odot dx^4$$

Since each $s_{i,j}$ was the component on the product $\omega^i \otimes \omega^j$, we can read off the basis of quadratic forms $\{\sigma^i\} = \mathfrak{G}$ and order them as we see fit:

$$\begin{aligned}
\sigma^1 &= \mathrm{e}^{x^3}\, dx^1 \odot dx^2 \\
\sigma^2 &= {x^1}^2\mathrm{e}^{2\,x^3}\, dx^2 \odot dx^2 + x^1\mathrm{e}^{2\,x^3}\, dx^2 \odot dx^4 + \mathrm{e}^{2\,x3}\, dx^4 \odot dx^4 \\
\sigma^3 &= x^1\mathrm{e}^{x^3}\, dx^2 \odot dx^3 + \mathrm{e}^{x^3}\, dx^3 \odot dx^4 \\
\sigma^4 &= dx^3 \odot dx^3
\end{aligned} \tag{5.3}$$

Observe that the number of basis quadratic forms σ^i on G/H is four whereas the number of basis quadratic forms $s_{i,j}\omega^i \otimes \omega^j$ on G was seven (as can be checked by counting the free variables s_{ij} in the matrix 5.2). To further demonstrate the algebraic nature of the above process of constructing Γ-invariant quadratic forms on G/H, we show how quickly the steps can be performed in Maple at the purely algebraic level. The following command initializes an abstract four-dimensional vector space:

$$> DGEnvironment[VectorSpace](4, V,\ vectorlabels\ = [E],\ formlabels = [tau]);$$

Observe that we chose the labels of the dual 1-forms to be given by τ. Then we compute the adjoint of e_5 (recall e_5 from Section 5.1 is the Maple representation of the isotropy h) or we may simply use the adjoint of Z from above. Let's restrict the adjoint $ad(Z)$ to the reductive complement.

$$> AdZr := LinearAlgebra[SubMatrix](AdZ, 1..4, 1..4);$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The next command generates all symmetric rank-2 tensors on V (we suppress its output):

$$> S := GenerateSymmetricTensors([tau1, tau2, tau3, tau4], 2):$$

$$[\tau^1 \otimes \tau^1, \frac{1}{2}\tau^1 \otimes \tau^2 + \frac{1}{2}\tau^2 \otimes \tau^1, \frac{1}{2}\tau^1 \otimes \tau^3 + \frac{1}{2}\tau^3 \otimes \tau^1, \frac{1}{2}\tau^1 \otimes \tau^4 + \frac{1}{2}\tau^4 \otimes \tau^1,$$
$$\tau^2 \otimes \tau^2, \frac{1}{2}\tau^2 \otimes \tau^3 + \frac{1}{2}\tau^3 \otimes \tau^2, \frac{1}{2}\tau^2 \otimes \tau^4 + \frac{1}{2}\tau^4 \otimes \tau^2, \tau^3 \otimes \tau^3, \frac{1}{2}\tau^3 \otimes \tau^4 + \frac{1}{2}\tau^4 \otimes \tau^3,$$
$$\tau^4 \otimes \tau^4]$$

Then we have enough information for the command *InvariantTensors* to generate the basis of ad(H)-invariant quadratic forms $\omega^i \otimes \omega^j$ on G. Note the command simply utilizes equation

(5.1):

$> absTensors := InvariantTensors([AdZr], S);$

$$absTensors := [\tau^1 \otimes \tau^1, \frac{1}{2}\tau^1 \otimes \tau^4 + \frac{1}{2}\tau^4 \otimes \tau^1, \frac{1}{2}\tau^2 \otimes \tau^3 + \frac{1}{2}\tau^3 \otimes \tau^2, \tau^4 \otimes \tau^4]$$

Note the abstract forms τ^i correspond to the duals ω^i of the left-invariant vector fields L. That is, if we replace each τ^i in $absTensors$ with an ω^i, the $absTensors$ become precisely the ad(H)-invariant quadratic forms $\omega^i \otimes \omega^j$ on G we drop to the σ^j on G/H under the local cross-section ι.

This concludes this portion of the example wherein we found a basis $\mathfrak{G}$ of Γ-invariant quadratic forms where Γ is an infinitesimal group action on G/H which at the preferred point $(0,0,0,0)$ determines the Lorentzian Lie algebra-subalgebra pair $[5,4,8]$ (found in Appendix A.1).

5.3 Residual diffeomorphism group

We wish to describe the process of gauge fixing or normalizing the most general metric $g \in \mathrm{span}(\mathfrak{G})$. Recall that $\mathfrak{G}$ denotes a basis of Γ-invariant quadratic forms on G/H and the span is taken over the ring of Γ-invariant functions. This normalizing process can allow us to remove certain extraneous parameters or functions from the local coordinate expression of g, returning an equivalent metric $\tilde{g}$ also having full isometry group G.

We define the *residual diffeomorphism group* to be the group of diffeomorphisms

$$\mathcal{R} = \{\phi : M \to M \mid \phi^*\sigma = \sigma, \forall \sigma \in \mathrm{span}(\mathfrak{G})\}.$$

Let $\mathfrak{X}(M)$ be the infinite dimensional Lie algebra of vector fields on a finite-dimensional differentiable manifold M. Suppose $\Gamma \subset \mathfrak{X}(M)$ is an r-dimensional Lie algebra of vector fields with basis $\{X_i\}$. Define the *normalizer* of Γ in $\mathfrak{X}(M)$ to be

$$\mathrm{Nor}(\Gamma, \mathcal{G}) = \{Z \in \mathfrak{X}(M) \mid [Z, X_i] = a_i^j X_j, \text{ for } i,j = 1 \ldots r, \text{ and } a_i^j \text{ constants}\}.$$

Note by definition of $\mathrm{Nor}(\Gamma, \mathfrak{X}(M))$ that Z may contain non-constant arbitrary Γ-invariant functions F. Suppose Γ is the finite-dimensional Lie algebra of Killing vectors for the general metric $g \in \text{span}(\mathfrak{G})$. Then given $Z \in \text{Nor}(\Gamma, \mathfrak{X}(M))$ and $X \in \Gamma$, we have

$$\begin{aligned} 0 &= \mathcal{L}_{[X,Z]}(g) \\ &= \mathcal{L}_X \mathcal{L}_Z(g) - \mathcal{L}_Z \mathcal{L}_X(g) \\ &= \mathcal{L}_X \mathcal{L}_Z(g) - \mathcal{L}_Z(0) \\ &= \mathcal{L}_X(\mathcal{L}_Z(g)). \end{aligned}$$

Thus $\mathcal{L}_Z g$ is invariant under all Killing vectors $X \in \Gamma$ and therefore $\mathcal{L}_Z g \in \text{span}(\mathfrak{G})$. That is, if $Z \in \text{Nor}(\Gamma, \mathfrak{X}(M))$ with flow ϕ_t, then $\phi_t \in \mathcal{R}$. It follows that if $\phi_t : M \to M$ is the local flow of Z and $\mathfrak{G} = \{\sigma^i\}$, then

$$\phi_t^*(\sigma^i) = A^i_j(t)\sigma^j \in \text{span}(\mathfrak{G}),$$

where the $A^i_j(t)$ depend on t, may contain the arbitrary functions F, and satisfy

$$X(A^i_j(t)) = 0,$$

for all $X \in \Gamma$. Hence under the flow of Z the general invariant metric $g = s_i\sigma^i$ becomes $\phi_t^*(g) = s_i A^i_j(t)\sigma^j = \tilde{s}_i\sigma^i$. The key point to be made here is that it may be possible to choose t or F in such a way to make $\tilde{s}_k = \pm 1$ or 0 (or any constant deemed convenient) for $\{\tilde{s}_k\} \subset \{\tilde{s}_i\}$. In this way we may be able to obtain an equivalent metric $\tilde{g}$ with fewer arbitrary parameters or functions, noting the general invariant metric can be obtained from the set $\{s_i\}$.

5.3.1 Example of homogeneous space continued

We continue the example of Section 5.2 now wishing to normalize the general invariant metric $g \in \text{span}(\mathfrak{G})$. First note that the Γ-invariant functions are constant functions only in this example. This is seen by solving $X(f) = 0$ for $f : M \to \mathbb{R}$ with X an arbitrary vector field in Γ and $M = G/H$. The general Γ-invariant metric g is then a linear combination of the quadratic forms $\mathfrak{G}$ in equation (5.3) over $\mathbb{R}$. Let t_1 through t_4 denote the constant coefficients of $g = t_i \sigma^i$:

$$g = t_1 e^{x^3}\, dx^1 \odot dx^2 + t_2 {x^1}^2 e^{2\,x^3}\, dx^2 \odot dx^2 + t_3 x^1 e^{x^3}\, dx^2 \odot dx^3$$
$$+\, t_2 x^1 e^{2\,x^3}\, dx^2 \odot dx^4 + t_4\, dx^3 \odot dx^3 + t_3 e^{x^3}\, dx^3 \odot dx^4 + t_2 e^{2\,x3}\, dx^4 \odot dx^4$$

Next we can compute the determinant of g (thinking of g as a symmetric matrix of functions) to see any conditions on the constants t_1 though t_4, keeping in mind that the determinant must be strictly negative. This can be done using the command *MetricDensity*. Note we simplify and factor the result:

$> factor(simplify(MetricDensity(g, 2)));$

$$-\frac{1}{16}\, e^{4\,x3}\, t1^2 \left(4\, t2\, t4 - t3^2\right)$$

Then to maintain a non-degenerate metric of Lorentzian signature, $t_1 \neq 0$,

$$4\, t_2\, t_4 - {t_3}^2 \geq 0,$$

and $t_2, t_4 \neq 0$.

To compute the normalizer of Γ in $\mathfrak{X}(M)$, we use the DifferentialGeometry command *InfinitesimalPseudoGroupNormalizer*:

$> PR := InfinitesimalPseudoGroupNormalizer(\Gamma,\ output = \text{“}list\text{”});$

$$[e^{-x3}\, \partial_{x4},\, \partial_{x3}]$$

It is easily verified in Maple that $[X, Z] = 0$ and therefore $[X, Z] \in \Gamma$ and that $\mathcal{L}_X(\mathcal{L}_Z g) = 0$ for all $X \in \Gamma$ and $Z \in PR$. We verify the latter statement in Maple:

$> LieDerivative(K, LieDerivative(PR[1], g));$

$$[0\, dx^1 \otimes dx^1, 0\, dx^1 \otimes dx^1, 0\, dx^1 \otimes dx^1, 0\, dx^1 \otimes dx^1, 0\, dx^1 \otimes dx^1]$$

and

$> LieDerivative(K, LieDerivative(PR[2], g));$

$$[0\,dx^1 \otimes dx^1, 0\,dx^1 \otimes dx^1, 0\,dx^1 \otimes dx^1, 0\,dx^1 \otimes dx^1, 0\,dx^1 \otimes dx^1]$$

Now we compute the flows of each $Z \in PR$ and compose the transformations, noting we provide the command *Flow* with a chosen name for flow parameter:

$> T1, T2 := Flow(PR[1], lambda1),\ Flow(PR[2], lambda2);$

$$T1, T2 := \mathit{x1} = \mathit{x1}, \mathit{x2} = \mathit{x2}, \mathit{x3} = \mathit{x3}, \mathit{x4} = \mathrm{e}^{-\mathit{x3}}\lambda 1 + \mathit{x4},$$

$\mathit{x1} = \mathit{x1}, \mathit{x2} = \mathit{x2}, \mathit{x3} = \lambda 2 + \mathit{x3}, \mathit{x4} = \mathit{x4}$

Let ϕ be the composition of maps $T1 \circ T2$:

$> phi := ComposeTransformations(T1, T2);$

$$\phi := \mathit{x1} = \mathit{x1}, \mathit{x2} = \mathit{x2}, \mathit{x3} = \lambda 2 + \mathit{x3}, \mathit{x4} = \mathrm{e}^{-\lambda 2 - \mathit{x3}}\lambda 1 + \mathit{x4}$$

We can pullback g under ϕ and determine which choices of λ_1 and λ_2 could help simplify its expression. Since we have composed the flows, we have

$$\phi^*(g) = \phi^*(t_i\sigma^i) = t_i A^i_j(\lambda_1, \lambda_2)\sigma^j \tag{5.4}$$

from the discussion above. We do this using the DifferentialGeometry commands *Pullback* with *GetComponents* to compute the components of the pulled back metric with respect to the basis *Glist* above. We use expand to clean the results:

$> lst := expand(GetComponents(Pullback(phi, G), Glist));$

$$[\mathit{t1}\,\mathrm{e}^{\lambda 2}, \mathit{t2}\left(\mathrm{e}^{\lambda 2}\right)^2, -2\,\mathrm{e}^{\lambda 2}\mathit{t2}\,\lambda 1 + \mathrm{e}^{\lambda 2}\mathit{t3}, \mathit{t2}\,\lambda 1^2 - \mathit{t3}\,\lambda 1 + \mathit{t4}]$$

Observe that this result is equivalent to computing equation (5.4), namely

$$tA\sigma = (t_1, t_2, t_3, t_4) \cdot A \cdot (\sigma^1, \sigma^2, \sigma^3, \sigma^4)^T,$$

where

$$A = (A^i_j) = \begin{bmatrix} e^{\lambda_2} & 0 & 0 & 0 \\ 0 & e^{2\lambda_2} & -2\lambda_1 e^{\lambda_2} & {\lambda_1}^2 \\ 0 & 0 & e^{\lambda_2} & -\lambda_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

From the above we see we can set the third expression in the list *lst*, namely

$$-2\,e^{\lambda 2} t_2\,\lambda_1 + e^{\lambda_2} t_3,$$

to zero and solve for λ_1, arriving at

$> sol1 := solve(lst[3], lambda1);$

$$sol1 := \left\{\lambda 1 = \frac{1}{2}\,\frac{t3}{t2}\right\}$$

Recall by the determinant of g, $t_2 \neq 0$. If we substitute this choice in to *lst*, we see the result to be 0 for the third component:

$> lst2 := expand(eval(lst, sol1));$

$$lst2 := \left[t1\;e^{\lambda 2},\, t2\,\left(e^{\lambda 2}\right)^2, 0, -\frac{1}{4}\,\frac{t3^2}{t2} + t4\right]$$

Now solve the first component for λ_2, getting the following:

$> sol2 := solve(lst2[1], lambda2);$

$$sol2 := \{\lambda 2 = -\ln(t1)\}$$

We know by the determinant that $t_1 \neq 0$, but we may have $t_1 < 0$. Thus choose $\{\lambda_2 = -\ln(|t1|)\}$. This gives the following:

$> lst3 := expand(eval(lst2, sol2));$

$$lst3 := \left[\frac{t1}{|t1|}, \frac{t2}{(|t1|)^2}, 0, -\frac{1}{4}\,\frac{t3^2}{t2} + t4\right]$$

Observe that for all $t_1 \neq 0$, $\frac{t1}{|t1|} \equiv \epsilon = \pm 1$. Then we may write the normalization as

$$[\epsilon, s_2, 0, s_4],$$

for some constants s_2 and s_4. Therefore the most general metric g is obtained from the following metric by the normalizer ϕ:

$$\frac{1}{2}\,\epsilon\,\mathrm{e}^{x^3}\,dx^1\,dx^2 + s_2\,{x^1}^2\mathrm{e}^{2\,x^3}\,dx^2\,dx^2 + s_2\,x^1\,\mathrm{e}^{2\,x^3}\,dx^2\,dx^4 + s_4\,dx^3\,dx^3 + s_2\,\mathrm{e}^{2\,x^3}\,dx^4\,dx^4.$$

Note this has been written in the more compact notation of the symmetric tensor product, namely

$$dx^i\,dx^j \equiv dx^i \odot dx^j = dx^i \otimes dx^j + dx^j \otimes dx^i.$$

We now summarize the key information gained from all parts of this example.

5.3.2 Example summary for homogeneous space

This section summarizes and illustrates the discussion above regarding the Lorentzian Lie algebra-subalgebra pair $[5, 4, 8]$. Contained in the summary are the following items:

1. a reference to where one may find the structure of the abstract Lie algebra in the literature;
2. the structure tables for the Lie algebra in one of two bases: (i) the "isotropy adapted" basis wherein the isotropy at the preferred point is the last vector field and the adjoint representation of the isotropy h of the abstract Lie algebra is in a standard form found in Table 5.1 and (ii) the "structure theory adapted" basis that manifests the structure of the Lie algebra as and can be identified in the literature referenced by item 1;
3. the isomorphisms used on the vector fields to get the tables in 2 (see Section 5.1);
4. the isotropy type and basis for the isotropy in the e_i basis (see Section 5.1);
5. the vector fields Γ constructed in Section 5.1;
6. the preferred point at which the isotropy is computed in the isotropy adapted basis;
7. the basis of Γ-invariant quadratic forms constructed in Section 5.2;

8. the determinant of the general invariant metric g and if applicable, the determinant of g restricted to the orbits;
9. the normalizer (or residual diffeomorphism) used to normalize or gauge fix the general metric in Section 5.3;
10. the components of the normalized metric in terms of the invariant quadratic forms of item 7 (see Section 5.3);
11. a reference number to the Petrov classification of spacetimes with symmetry [2] where the spacetimes of this classification can be found.

[5, 4, 8]

1. Reference : (F13, 6), Rozum

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	.	$-Y_1$	.	.	.
Y_2		.	$\frac{1}{2}Y_3 - \frac{1}{2}Y_4$	$-\frac{1}{2}Y_3 + \frac{1}{2}Y_4$	.
Y_3			.	$2\,Y_1$	$-Y_4$
Y_4				.	$-Y_3$
Y_5					.

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	$-e_1$	.
e_2		.	e_1	.	e_2
e_3			.	$-e_3$	$-e_3$
e_4				.	.
e_5					.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to 1/2\,Y_3 + 1/2\,Y_4, X_3 \to -1/2\,Y_3 + 1/2\,Y_4, X_4 \to Y_2, X_5 \to -Y_5]$

$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4, X_5 \to e_5]$

4. Isotropy: F13 $[\,e_5]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^4} \qquad X_4 = -x^2\,\partial_{x^2} + \partial_{x^3} - x^4\,\partial_{x^4}$$
$$X_2 = \partial_{x^1} - x^2\,\partial_{x^4} \qquad X_5 = x^e\,\partial_{x^1} - x^2\,\partial_{x^2}$$
$$X_3 = \partial_{x^2}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = e^{x^3}\, dx^1\, dx^2$$
$$\sigma^2 = {x^1}^2 e^{2\,x^3}\, dx^2\, dx^2 + x^e e^{2\,x^3}\, dx^2\, dx^4 + e^{2\,x^3}\, dx^4\, dx^4$$
$$\sigma^3 = x^1 e^{x^3}\, dx^2\, dx^3 + e^{x^3}\, dx^3\, dx^4$$
$$\sigma^4 = dx^3\, dx^3$$

8. Determinants : $\det(g) = -1/16\, {s_1}^2 e^{4\,x^3} \left(4\, s_2 s_4 - {s_3}^2\right)$

9. Normalizers:

$$\Phi_1 = [x^e = x^e, x^2 = x^2, x^3 = x^3 + \lambda_2, x^4 = \lambda_1 e^{-x^3 - \lambda_2} + x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$$[[\epsilon, s_2, 0, s_4], [\epsilon^2 = 1]]$$

11. Petrov Reference: $[33, 21, 0]$

The above information has been computed and likewise summarized for the Lorentzian Lie algebra-subalgebra pairs of Chapter 3 and can be found in Appendix A.2. We next give an example of normalizing a general invariant metric for a simple G space consisting of a four-dimensional Lie algebra of vector fields with one-dimensional isotropy.

5.3.3 Example of Simple G space

The previous example found in sections 5.1, 5.2, and 5.3, was a case of a five-dimensional isometry group G acting transitively on a four-dimensional space. We wish to now give an example of normalizing a general four-dimensional Lorentzian metric whose Killing algebra is a four-dimensional Lie algebra of vector fields acting on three-dimensional orbits. This is a case in which the residual diffeomorphisms depend on arbitrary functions.

Using the methods outlined in sections 5.1 and 5.2, we can construct the following vector field system which at the origin gives the Lorentzian pair $[4,3,7]$ found in Appendix A.1:

$$\Gamma = [\,\partial_{x^2},\, \partial_{x^3}, -\partial_{x^1} + x^2\,\partial_{x^2} + x^3\,\partial_{x^3}, -x^3\,\partial_{x^2} + x^2\,\partial_{x^3}]$$

Note these vector fields have one-dimensional isotropy at the origin and act on three-dimensional orbits. Observe that the vector field $F(x^4)\partial_{x^4}$ (with $F(x^4)$ arbitrary) commutes with all $X \in \Gamma$ and is transverse to the orbits. Thus Γ defines a simple G space on M.

We now initialize a *four*-dimensional manifold M and the vector fields defining Γ in Maple:

$> DGsetup([x1, x2, x3, x4], M):$

$> \Gamma := [D\_x2,\ D\_x3, -D\_x1 + x2 * D\_x2 + x3 * D\_x3, -x3 * D\_x2 + x2 * D\_x3]$

$$\Gamma := [\,\partial_{x^2},\, \partial_{x^3}, -\partial_{x^1} + x2\,\partial_{x^2} + x3\,\partial_{x^3}, -x3\,\partial_{x^2} + x2\,\partial_{x^3}]$$

The invariant quadratic forms on M (see Section 5.2) are

$> Glist := [\,dx1 \,\&t\ dx1, 1/2\,dx1 \,\&t\ dx4 + 1/2\,dx4 \,\&t\ dx1, \mathrm{e}^{2\,x1}\,dx2 \,\&t\ dx2$
$+\,\mathrm{e}^{2\,x1}\,dx3 \,\&t\ dx3,\ dx4 \,\&t\ dx4]$

$$Glist := \left[\,dx^1 \otimes dx^1, \frac{1}{2}\,dx^1 \otimes dx^4 + \frac{1}{2}\,dx^4 \otimes dx^1, \mathrm{e}^{2\,x1}\,dx^2 \otimes dx^2\right.$$
$$\left. +\,\mathrm{e}^{2\,x^1}\,dx^3 \otimes dx^3,\ dx^4 \otimes dx^4\right]$$

The Γ-invariant functions are functions of the coordinate x^4. Then we can use the routine *DGzip* to form the most general metric tensor g from *Glist* using the arbitrary functions $t_1(x^4)$, $t_2(x^4)$, $t_3(x^4)$, and $t_4(x^4)$ of coordinate x^4 as coefficients:

$> g := DGzip([t1(x4), t2(x4), t3(x4), t4(x4)], Glist)$

$$g := t1\,(x4)\;dx^1 \otimes dx^1 + \frac{1}{2}\,t2\,(x4)\;dx^1 \otimes dx^4 + t3\,(x4)\,\mathrm{e}^{2\,x1}\,dx^2 \otimes dx^2$$

$$+ t3\,(x4)\,e^{2\,x1}\,dx^3 \otimes dx^3 + \frac{1}{2}\,t2\,(x4)\,dx^4 \otimes dx^1 + t4\,(x4)\,dx^4 \otimes dx^4$$

Note $\mathcal{L}_X g = 0$ for all $X \in \Gamma$ by construction. We can compute the determinant of g (thinking of g as a symmetric matrix of functions $t_i(x^4)$ and noting that the determinant must be negative and non-zero) to find conditions on the function coefficients $t_i(x^4)$. This can be done using the command *MetricDensity*. We simplify and factor the result:

$> factor(simplify(MetricDensity(g, 2)));$

$$1/4\,(t3\,(x4))^2\,e^{4\,x1}\left(4\,t1\,(x4)\,t4\,(x4) - (t2\,(x4))^2\right)$$

Therefore, to maintain a non-degenerate metric of Lorentzian signature, we must have $t_3 \neq 0$ and

$$4\,t_1\left(x^4\right)t_4\left(x^4\right) - t_2\left(x^4\right)^2 < 0.$$

However, $t_1(x^4)$ can be zero in which case the metric would be null on the orbits. This is seen by computing the determinant of the metric restricted to the orbits. The matrix representation of g restricted to the orbits can be found in Maple by the command *TensorInnerProduct*. The orbits are given by the first three vector fields in the basis above defining Γ and we'll call them V:

$> V := [\,\Gamma[1], \Gamma[2], \Gamma[3]\,];$

$$V := [\,\partial_{x2}, \partial_{x3}, -\partial_{x1} + x2\,\partial_{x2} + x3\,\partial_{x3}]$$

Then the matrix representation of g restricted to the orbits is

$> A := TensorInnerProduct(g, V, V);$

$$\begin{bmatrix} t3\,(x4)\,e^{2\,x1} & 0 & x2\,t3\,(x4)\,e^{2\,x1} \\ 0 & t3\,(x4)\,e^{2\,x1} & x3\,t3\,(x4)\,e^{2\,x1} \\ x2\,t3\,(x4)\,e^{2\,x1} & x3\,t3\,(x4)\,e^{2\,x1} & t1\,(x4) + x2^2 t3\,(x4)\,e^{2\,x1} + x3^2 t3\,(x4)\,e^{2\,x1} \end{bmatrix},$$

and its determinant is

$> LinearAlgebra[Determinant](A);$

$$\_t1\,(x4)\,(\_t3\,(x4))^2\,\left(\mathrm{e}^{2\,x1}\right)^2,$$

which is clearly zero when $t_1(x^4) = 0$. We will begin under the assumption that $t_1(x^4) \neq 0$, then consider the case $t_1(x^4) = 0$ subsequently. To compute the normalizer of Γ in $\mathfrak{X}(M)$, we use the DifferentialGeometry command *InfinitesimalPseudoGroupNormalizer*:

$> PR := InfinitesimalPseudoGroupNormalizer(\Gamma,\ output = \text{“}list\text{”});$

$$[\_F1\,(x4)\ \partial_{x^4}, \_F2\,(x4)\ \partial_{x^1}]$$

Observe that the output contains two vector fields with arbitrary functions of x^4 as coefficients. Then the vector fields are (respectively) thought of as the infinitesimal generators of the following transformations in the residual diffeomorphism group:

$> phi1 := Transformation(M, M, [x1 = x1,\ x2 = x2,\ x3 = x3,\ x4 = A(x4)]);$

$$\phi 1 := x1 = x1,\ x2 = x2,\ x3 = x3\ , x4 = A(x4)$$

$> phi2 := Transformation(M, M, [x1 = x1, x2 = x2, x3 = x3, x4 = A(x4)]);$

$$\phi 2 := x1 = x1 + B(x4),\ x2 = x2,\ x3 = x3\ , x4 = x4$$

where $A(x4)$ and $B(x4)$ are arbitrary functions of x^4. We can pullback g under $\phi 1$ and $\phi 2$ and determine appropriate choices of $A(x4)$ and $B(x4)$ to give a normalized equivalent metric. We do this using the DifferentialGeometry commands *Pullback* in conjuction with *GetComponents* to compute the components of the pulled back metric with respect to the basis *Glist* above. We use expand to clean the results. Using $\phi 2$, we get

$> lst := expand(\,GetComponents(\,Pullback(\,phi2,\,G),\,Glist));$

$$[t1(x4), 2\ t1(x4) * (\mathrm{diff}(B(x4), x4)) + t2(x4), t3(x4) * \mathrm{e}^{2*B(x4)},$$
$$t1(x4) * (\mathrm{diff}(B(x4), x4))^2 + t2(x4) * (\mathrm{diff}(B(x4), x4)) + t4(x4)]$$

noting that $\operatorname{diff}(B(x4), x4) \equiv \frac{d}{dx^4}B(x^4)$. From the above we see we can set the second expression in the list, namely $2t_1(x^4)\frac{d}{dx^4}B(x^4) + t_2(x^4)$, to zero and solve for $\frac{d}{dx^4}B(x^4)$, arriving at the following:

$> sol1 := solve(lst[2], \operatorname{diff}(B(x4), x4));$

$$sol1 := \frac{d}{dx4}B(x4) = -\frac{1}{2}\frac{t2(x4)}{t1(x4)}$$

Recall by the determinant of g, $t_1 \neq 0$. If we substitute this choice in to lst, we see the result to be 0 for the second component:

$> lst2 := expand(eval(lst, sol1));$

$$lst2 := \left[t1\,(x4)\,, 0, t3\,(x4)\,\mathrm{e}^{2\,B(x4)}, -\frac{1}{4}\frac{(t2\,(x4))^2}{t1\,(x4)} + t4\,(x4)\right]$$

Repeating this procedure using the map $\phi 1$ for pullback, we get as fourth component (after pullback)

$$\frac{\left(4t_1(A(x^4))t_4(A(x^4)) - t_2(A(x^4))^2\right)\left(\frac{d}{dx^4}A(x^4)\right)^2}{4t_1(A(x^4))}.$$

Now set this expression equal to 1 and solve for $\frac{d}{dx^4}A(x^4)$:

$$\frac{d}{dx^4}A(x^4) = \pm\frac{2\sqrt{(4t_1(A(x^4))t_4(A(x^4)) - t_2(A(x^4))^2)t_1(A(x^4))}}{(4t_1(A(x^4))t_4(A(x^4)) - t_2(A(x^4))^2)}$$

Choose

$$\frac{d}{dx^4}A(x^4) = \frac{2\sqrt{|(4t_1(A(x^4))t_4(A(x^4)) - t_2(A(x^4))^2)t_1(A(x^4))|}}{(4t_1(A(x^4))t_4(A(x^4)) - t_2(A(x^4))^2)}.$$

This choice gives after pullback by $\phi 1$ the components as

$$\left[s_1(x^4), 0, s_3(x^4), \frac{|(4t_1(A(x^4))t_4(A(x^4)) - t_2(A(x^4))^2)t_1(A(x^4))|}{(4t_1(A(x^4))t_4(A(x^4)) - t_2(A(x^4))^2)t_1(A(x^4))}\right],$$

where we've relabeled the first and third components to $s_1(x^4)$ and $s_3(x^4)$ respectively as these components are comprised of combinations of the arbitrary functions $t_i(x^4)$. By the determinant (metric density) of g and the assumption that $t_1(x^4) \neq 0$, we can rewrite this list of components as $\left[s_1(x^4), 0, s_3(x^4), \epsilon\right]$, where $\epsilon = \pm 1$. Then the most general invariant metric g can be gauged or normalized to

$$\tilde{g} = s_1\left(x^4\right)\, dx^1\, dx^1 + s_3\left(x^4\right) \mathrm{e}^{2\,x^1}\, dx^2\, dx^2 + s_3\left(x^4\right) \mathrm{e}^{2\,x^1}\, dx^3\, dx^3 + \epsilon\, dx^4\, dx^4.$$

It's easily checked in Maple that $\mathcal{L}_X \tilde{g} = 0$ for all $X \in \Gamma$ and that Γ comprises the full Killing algebra using the command *KillingVectors*.

Now, we must briefly consider the null case $t_1(x^4) = 0$. The determinant under this assumption is

$$-\frac{1}{4} t_2(x^4)^2 t_3(x^4)^2 \mathrm{e}^{4x^1}.$$

By choosing

$$\frac{d}{dx4} B(x^4) = \frac{-t_4(x^4)}{t_2(x^4)}$$

and

$$\frac{d}{dx4} A(x^4) = \frac{2}{t_2(A(x^4))},$$

we have the components $[0, 2, s_3(x4), 0]$, where the third component has been relabeled. Thus the final normalized metric in this case is

$$\tilde{g} = dx^1\, dx^4 + s_3(x^4) \mathrm{e}^{2\,x1}\, dx^2\, dx^2 + s_3(x^4) \mathrm{e}^{2\,x^1}\, dx^3\, dx^3 + dx^4\, dx^1.$$

Again, one easily checks in Maple that $\mathcal{L}_X \tilde{g} = 0$ for all $X \in \Gamma$ and that Γ comprises the full Killing algebra using the command *KillingVectors*.

This entry, $[4, 3, 7]$, along with all other spacetimes associated to the Lorentzian pairs classified in Chapter 3 can be found in Appendix A.2.

5.4 Lorentzian Pairs $[6, 4, 6]$ and $[7, 4, 5]$

In this section we address two special cases of Lorentzian pairs in the classification given in Chapter 3 of this dissertation, namely pairs $[6, 4, 6]$ and $[7, 4, 5]$.

First, recall the pair $[6, 4, 6]$, consisting of a six-dimensional Lie algebra and two-dimensional subalgebra with isotropy type $F10$ and five parameters in the structure equations:

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	.	.	.	$-ae_1$
e_2		.	.	e_1	.	$be_3 - e_4$
e_3			.	.	e_1	$-e_5$
e_4				.	.	$ce_2 + de_3 - ae_4$
e_5					.	$de_2 + fe_3 - be_4 - ae_5$
e_6						.

This pair corresponds to (33.54) in Petrov's classification [2] as can be seen by noting the nilradical is $\mathfrak{n}_{5,3}$. However, the structure equations on the sixth vector do not match those of $[6,4,6]$ so a change of basis is needed to find vector fields whose abstract Lie algebra coincides with $[6,4,6]$. The following vector fields, taken from Petrov and slightly rearranged, give the desired nilradical, noting the isotropy is given by X_2 and X_3:

$$X_1 = -\,\partial_{x^1}$$
$$X_2 = x^2\,\partial_{x^1} + \phi\left(x^4\right)\,\partial_{x^2} + \psi\left(x^4\right)\,\partial_{x^3}$$
$$X_3 = x^3\,\partial_{x^1} + \psi\left(x^4\right)\,\partial_{x^2} + \theta\left(x^4\right)\,\partial_{x^3}$$
$$X_4 = \,\partial_{x^2}$$
$$X_5 = \,\partial_{x^3}$$

Note the arbitrary functions $\phi\left(x^4\right)$, $\psi\left(x^4\right)$, and $\theta\left(x^4\right)$. Define a vector field $X_6 = A^i(x^1, x^2, x^3, x^4)\partial_{x^i}$. Computing the Lie brackets $[X_6, X_i]$, $i = 1..5$, demanding they match abstractly those of $[6,4,6]$, and solving the resulting PDE for the unknown functions A^i, we have

$$\begin{aligned} X_6 = {} & (-ax^1 + \frac{1}{2}\,c{x^2}^2 + dx^3\,x^2 + \frac{1}{2}\,f{x^3}^2)\,\partial_{x^1} \\ & + (\phi\left(x^4\right)dx^3 + \phi\left(x^4\right)cx^2 + d\psi\left(x^4\right)x^2 + f\psi\left(x^4\right)x^3 - bx^3 - x^2\,a)\,\partial_{x^2} \\ & + (\left(dx^2 + fx^3\right)\theta\left(x^4\right) + \left(cx^2 + dx^3\right)\psi\left(x^4\right) - x^3\,a)\,\partial_{x^3} + \,\partial_{x^4}, \end{aligned}$$

with a, b, c, d, f the parameters of the structure equations of $[6,4,6]$. To solve the system, we placed conditions on the functions $\phi\left(x^4\right)$, $\psi\left(x^4\right)$, and $\theta\left(x^4\right)$ given by the following

ODE:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x^4}\phi\left(x^4\right) &= a\phi\left(x^4\right)-\phi\left(x^4\right)^2-\psi\left(x^4\right)^2-c,\\
\frac{\mathrm{d}}{\mathrm{d}x^4}\psi\left(x^4\right) &= a\psi\left(x^4\right)-\psi\left(x^4\right)\phi\left(x^4\right)+b\phi\left(x^4\right)-\theta\left(x^4\right)\psi\left(x^4\right)-d,\\
\frac{\mathrm{d}}{\mathrm{d}x^4}\theta\left(x^4\right) &= a\theta\left(x^4\right)-\psi\left(x^4\right)^2+2\,b\psi\left(x^4\right)-\theta\left(x^4\right)^2-f
\end{aligned}
$$

subject to the initial conditions demanded by the isotropy. The Killing equations $\mathcal{L}_{X_i}g=0$, $i=1..6$, with $g=g_{\alpha\beta}dx^\alpha\otimes dx^\beta$, can be solved for functions $g_{\alpha\beta}$ of the coordinates. This gives the following metric tensor:

$$
\begin{aligned}
g = &-\frac{\mathrm{e}^{\omega}\left(\left(\frac{\mathrm{d}}{\mathrm{d}x^4}\theta\left(x^4\right)\right)\frac{\mathrm{d}}{\mathrm{d}x^4}\phi\left(x^4\right)-\left(\frac{\mathrm{d}}{\mathrm{d}x^4}\psi\left(x^4\right)\right)^2\right)}{\frac{\mathrm{d}}{\mathrm{d}x^4}\phi\left(x^4\right)}\,dx^1\,dx^4+\frac{\left(\frac{\mathrm{d}}{\mathrm{d}x^4}\theta\left(x^4\right)\right)\mathrm{e}^{\omega}}{\frac{\mathrm{d}}{\mathrm{d}x^4}\phi\left(x^4\right)}\,dx^2\,dx^2\\
&-\frac{\left(\frac{\mathrm{d}}{\mathrm{d}x^4}\psi\left(x^4\right)\right)\mathrm{e}^{\omega}}{\frac{\mathrm{d}}{\mathrm{d}x^4}\phi\left(x^4\right)}\,dx^2\,dx^3-\frac{\left(\frac{\mathrm{d}}{\mathrm{d}x^4}\psi\left(x^4\right)\right)\mathrm{e}^{\omega}}{\frac{\mathrm{d}}{\mathrm{d}x^4}\phi\left(x^4\right)}\,dx^3\,dx^2+\mathrm{e}^{\omega}\,dx^3\,dx^3\\
&-\frac{\mathrm{e}^{\omega}\left(\left(\frac{\mathrm{d}}{\mathrm{d}x^4}\theta\left(x^4\right)\right)\frac{\mathrm{d}}{\mathrm{d}x^4}\phi\left(x^4\right)-\left(\frac{\mathrm{d}}{\mathrm{d}x^4}\psi\left(x^4\right)\right)^2\right)}{\frac{\mathrm{d}}{\mathrm{d}x^4}\phi\left(x^4\right)}\,dx^4\,dx^1+k\,dx^4\,dx^4
\end{aligned}
$$

where

$$
\omega=-2\int\frac{A\psi\left(x^4\right)^2+B\psi\left(x^4\right)+C\theta\left(x^4\right)}{\frac{\mathrm{d}}{\mathrm{d}x^4}\phi\left(x^4\right)}\,\mathrm{d}x^4
$$

with

$$
A=-\phi\left(x^4\right)+a,\;\; B=b\phi\left(x^4\right)-d,\;\; C=\phi\left(x^4\right)^2-a\phi\left(x^4\right)+c,\;\; k=\text{const}.
$$

Efforts along these same lines to find a vector field system giving the family of Lorentzian Lie algebra pairs $[7,4,5]$ have not succeeded. Other methods beyond those described here will likely need to be employed to do so. However, the classifier *HomogeneousSpaceClassifier* is still able to classify the symmetry algebras of metrics having Killing algebras as special cases of $[7,4,5]$ (and likewise for $[6,4,6]$). No vector field systems or metrics at this time will be stored in the database for either $[6,4,6]$ or $[7,4,5]$ until they can be split into manageable cases in the future.

5.5 Conclusion

With the noted exceptions of $[6,4,6]$ and $[7,4,5]$, all Lorentzian Lie algebra-subalgebra pairs discovered in Chapter 3 have been associated to spacetimes (M, g) using the methods described in this chapter. The vector fields, invariant quadratic forms, and normalizations of these spacetimes, that is, the classification of spacetimes with symmetry, can be found in Appendix A.2.

We emphasize that this classification of spacetimes with symmetry is in one-to-one correspondence with the Lorentzian Lie algebra-subalgebra pairs of Chapter 3 (found in Appendix A.1). Therefore the software of Chapter 4 called *SpacetimeSymmetryClassifier*, which was adapted from the program *HomogeneousSpaceClassifier*, will uniquely associate to any (simple G) spacetime having isometry dimension between three and seven (inclusive) a Lorentzian pair of Chapter 3.

Lastly, note that the Lorentzian pairs with trivial isotropy (three and four dimensional Lie algebras) do not have normalizations in Appendix A.1. The normalizing for these cases requires a fair amount of branching and is left as a future project. Thus there will be seen an absence of determinants and normalizers in the presentation of results given in Appendix A.1 for the Lorentzian pairs with trivial isotropy.

CHAPTER 6

PETROV'S CLASSIFICATION OF SPACETIMES WITH SYMMETRY

One of the primary motivations for this dissertation was to provide an independent verification of Petrov's classification of spacetimes with symmetry given in the book *Einstein Spaces* [2]. In this chapter we will

i) identify and correct typos and small errors in Petrov;
ii) identify Petrov entries for which the Killing vector field systems are diffeomorphic and give explicit diffeomorphisms;
iii) identify Petrov entries for which the the given metric is not the most general invariant metric, allowing for proper normalization;
iv) identify Petrov entries for which the Killing vector field system for the given metric is larger than that provided by Petrov;
v) identify the non-reductive entries in Petrov;
vi) identify non-simple G Killing vector field systems in Petrov;
vii) give the symmetry classification of each simple G entry in Petrov using software from Chapter 4;
viii) identify the reductive Lorentzian pairs from Chapter 3 and the non-reductive Lorentzian pairs from Fels [5] which do not appear in Petrov.

Entries in Petrov's classification consist of a gauge fixed metric and a corresponding Lie algebra of Killing vectors. We deem an entry in Petrov correct if the general invariant metric can be gauge fixed to the metric given by Petrov and if the given Killing vectors are precisely those of the given metric. Those entries which are correct (and in which the vector field systems are not diffeomorphic to others in the classification) will not be discussed in the commentary that follows in Section 6.1.

With regards to ii), for vector field systems in Petrov which are diffeomorphic, one should check that the respective metrics are inequivalent. This is true for many entries simply because the orbit type of the Killing vectors has changed. Indeed, Petrov consistently splits the classification according to whether or not the metric is null on the orbits. In the case of null orbits, Petrov defines a *special operator* to be a null Killing vector field which

Table 6.1: The G_3 on V_3 for which it is unresolved whether or not the metrics are inequivalent.

Petrov Entry	Admits Special Operator
(31.28) and (31.29)	No
(31.32) and (31.33)	Yes
(31.34) and (31.35)	No
(31.39) $\epsilon = 0, q \neq 1$ and (31.42)	No
(31.43) and (31.44)	No

is orthogonal to all other Killing vector fields. However, there are a number of entries in Petrov where the Killing vector field systems admit a special operator and are diffeomorphic but efforts to determine if the metrics as given are equivalent have not been successful. A similar statement can be made for diffeomorphic systems not admitting a special operator. These cases are restricted to G_3 on V_3 and are listed in Table 6.1.

Regarding iii), the techniques of Chapter 5 concerning normalizing (or gauge fixing) of general invariant metrics were applied to all cases of G_3 on V_2, G_4 on V_3, and G_5 on V_4. *We emphasize that verification of gauge fixing for G_3 on V_3 and G_4 on V_4 was not performed.* For those spacetimes in Appendix A.2 which appear in Petrov, we then compared our normalizations against those of Petrov. As seen in the commentary below, there are many instances where Petrov *over* or *under* normalized the metrics. By over normalize we mean that the metric given by Petrov does not correspond to the most general invariant metric. By under normalize, we mean we were able to further normalize the metric and eliminate more functions and constants than that given by Petrov. Over normalization is an incorrect result whereas under normalization is merely an oversight.

We give an example to illustrate what is at issue. The Gödel spacetime is a homogeneous space admitting a G_5 on a V_4 and can be found in Stephani et al. [1] as (12.26). A basis of Killing vectors for the Gödel spacetime is given by the following vector fields,

$$K_{\text{Gödel}} = \left[-2\partial_z, -\partial_x + z\partial_z, 2\mathrm{e}^{-x}\partial_t - z\partial_x + \left(\frac{z^2}{2} - \mathrm{e}^{-2x}\right)\partial_z, -\partial_t, \partial_y\right].$$

Using the software of this dissertation, we can give the symmetry classification of Gödel as $[5, 4, 1]$ (see Chapter 5). The only entry in Petrov with the symmetry classification of $[5, 4, 1]$ is (33.17) $\epsilon = -1$. However, there is a scalar invariant that can be used to separate the Gödel and Petrov metrics. The Petrov Killing vectors for this entry can be written

$$K_{\text{Petrov}} = [\sqrt{2}\partial_{x^2}, x^2\partial_{x^2} + \partial_{x^3}, \frac{\sqrt{2}\mathrm{e}^{x^3}}{2}\partial_{x^1} + \frac{\sqrt{2}}{2}(\mathrm{e}^{2x^3} - x^{2^2})\partial_{x^2} - \sqrt{2}\ x^2\partial_{x^3}, -\frac{\sqrt{2}}{4}\partial_{x^1}, -\partial_{x^4}].$$

The following map ψ shows the Killing vectors of Petrov and Gödel are diffeomorphic (meaning ψ and its inverse are used to pushforward one set of Killing vectors to the other)

$$\psi = \left[t = 2\sqrt{2}\ x^1 + c_1, x = -x^3, y = -x^4 + c_2, z = -x^2\ \sqrt{2}\right],$$

where c_1 and c_2 are arbitrary constants. The general invariant metric of (33.17) $\epsilon = -1$ has the form

$$g = s_1\sigma_1 + s_2\sigma_2 + s_3\sigma_3 + s_4\sigma_4,$$

where the components $[s_1, s_2, s_3, s_4]$ are constants such that $4s_1s_4 - 4s_2s_4 - s_3^2 \neq 0$ (since the case $4s_1s_4 - 4s_2s_4 - s_3^2 = 0$ gives a degenerate metric). We can gauge fix (or normalize) this general invariant metric by pullback using the residual diffeomorphism

$$\phi = [x^1 = \xi_2 x^4 + x^1, x^2 = x^2, x^3 = x^3, x^4 = \xi_1 x^4],$$

and judicious use of parameters ξ_1 and ξ_2. Note that ψ is the composition of the flows of vector fields in the normalizer of K_{Petrov} in $\mathfrak{X}(M)$ (see Section 5.3). Assuming $s_1 \neq s_2$ and choosing parameter values

$$\xi_1 = \frac{2|(4s_1s_4 - 4s_2s_4 - s_3^2)(s_1 - s_2)|^{1/2}}{(4s_1s_4 - 4s_2s_4 - s_3^2)},$$

$$\xi_2 = \frac{s_3|(4s_1s_4 - 4s_2s_4 - s_3^2)(s_1 - s_2)|^{1/2}}{2(4s_1s_4 - 4s_2s_4 - s_3^2)(s_1 - s_2)}$$

gives components $[\tilde{s}_1, \tilde{s}_2, 0, e]$, where $e = \pm 1$, and $\tilde{s}_1$ and $\tilde{s}_2$ are arbitrary. If we instead assume $s_1 = s_2$, and note that this implies $s_3 \neq 0$ since if $s_3 = 0$ the metric is then degenerate, we may choose

$$\xi_1 = \frac{-2}{s_3}$$

and

$$\xi_2 = -\frac{s_4}{s_3^2},$$

to get components $[\tilde{s}_1, \tilde{s}_1, -2, 0]$. However, Petrov gives only the second normalization. The Gödel metric corresponds to the first normalization. These metrics are invariantly characterized by the Riemann invariant

$$m_4 = -\frac{1}{8}\bar{C}_{acdb}S^{cd}\overset{+}{C}{}^{b}{}_{efg}S^{ef}S^{ag},$$

where C_{abcd} is the Weyl tensor, S_{ab} the trace-free Ricci tensor, $\overset{+}{C}_{abcd} = \frac{1}{2}(C_{abcd} - i{*}C_{abcd})$ the self-dual Weyl tensor with ${*}C_{abcd} \equiv \frac{1}{2}\epsilon_{abef}C^{ef}{}_{cd}$ and where $\bar{C}_{abcd}$ is the complex conjugate of $\overset{+}{C}_{abcd}$ and ϵ_{abef} the permutation symbols or Levi-Civita symbols (see Carminati [23], Stephani et al. [1] and Hicks [24] for further details). For Petrov's given metric in this example, $m_4 = 0$ whereas m_4 cannot be zero for the Gödel metric. This is an example of a type of over normalizing found in Petrov. Note that all discussion of components in Section 6.1 refers to the components of the general invariant metric in the basis $\mathfrak{G}$ as described in this example.

Regarding iv), there are occurrences in Petrov for which the general invariant metric of Petrov's given Killing vectors admits additional Killing vectors beyond those provided by Petrov. For instance, this is the case for Petrov's entry (32.18). Petrov claims (32.18) to be a G_4 acting on a V_3. However, the general invariant metric admits two additional Killing vectors not provided by Petrov, giving a G_6 acting on a V_3. The entries in Petrov admitting additional Killing vectors are listed in Table 6.2.

Table 6.2: Petrov entries admitting additional symmetries

(32.18)
(32.20)
(32.27)
(33.24)
(33.25)
(33.26)
(33.27)
(33.38)
(33.39)

Table 6.3: Non-reductive Lorentzian pairs in Petrov

(33.5)
(33.6)
(33.7)
(33.8)
(33.12)
(33.13)
(33.29)
(33.30)

For v), observe Table 6.3, which identifies the non-reductive simple G entries in Petrov's classification.

In regards to vi), note that Petrov's classification includes non-simple G spaces. We identify those non-simple G spaces in Petrov in Table 6.4. See example 2.4.6 for more information on identifying such spaces.

For vii), see Section 6.2 wherein we give the symmetry classification of all simple G entries in Petrov using the software of Chapter 4.

Regarding viii), it's important to note that there are spacetimes in the classification of Chapter 5 of this dissertation that are not found in the Petrov classification. These can be described in one of the following two ways.

a) Those Lorentzian Lie algebra-subalgebra pairs of Chapter 3 missing entirely from Petrov, and
b) for a given Lorentzian pair in Petrov, the metrics may be incorrectly normalized.

Table 6.4: The spacetimes missing from Petrov listed together with the non-Lorentzian entries in Petrov and the non-simple G entries of Petrov.

Spacetimes missing from Petrov	Non-Lorentzian Petrov Entries	Non-simple G Petrov Entries
$[5, 4, -1]$	(32.21)	(30.8)
$[6, 3, 1]$	(32.23) $e_2 = 1$	(32.18)
$[6, 3, 4]$	(32.24) $e_2 = 1$	(32.20)
$[6, 4, -1]$	(33.36)	(32.26)
$[7, 4, 3]$	(33.37)	(33.40) (C)
$[7, 4, 4]$		

We emphasize that *every entry in Petrov is covered by our classification...*

- except the non-simple G spaces in Table 6.4
- and those entires of non-Lorentzian signature also in Table 6.4.

Finally let us stress that to complete an independent verification of Petrov's classification we need to 1) perform gauge fixing on the remaining cases and 2) classify non-simple G Lorentzian spaces.

Following the commentary, we will display in Section 6.2 tables giving the symmetry classification of Petrov's entries using the classifier of Chapter 4.

6.1 Commentary on Petrov

Petrov denotes by $\overset{*}{V}_n$ those n-dimensional orbits whose induced metric is degenerate (null orbit type). Otherwise the manifold is denoted V_n. Note that all discussion of components s_i below refers to the components of the general invariant metric in the basis $\mathfrak{G}$ found in Appendix A.2.

6.1.1 G_3 on V_2

(30.1): The gauge fixing provided by Petrov is incomplete in that he fails to consider the case in which the fourth component can be zero. The Petrov given normalization depends on $s_4 \neq 0$. For $s_4 = 0$ we are able to reduce the second (and fourth) component to 0 and the third component to 2.

(30.3) *Type IV*: No explicit metric was given for this space. Killing vectors are found at (30.3) type IV. This is the space Bowers [4] claimed was not in Petrov. The gauge fixing provided by Petrov is incomplete in that he fails to consider the case in which the fourth component can be zero. The Petrov given normalization depends on $s_4 \neq 0$. For $s_4 = 0$ we are able to reduce the second and fourth components to 0 and the third component to 2. Note that an invariant has not been found which distinguishes these normalizations.

(30.4), (30.5): As Lorentzian Lie algebra-subalgebra pairs, (30.4) and (30.5) are equivalent. By a simple change of basis on (30.4), namely $\{X_2, X_1, X_3\}$, one sees the structure equations are identical to (30.5).

(30.6): The gauge fixing provided by Petrov is incomplete in that he fails to consider the case in which the fourth component can be zero. The Petrov given normalization depends on $s_4 \neq 0$. For $s_4 = 0$ we were able to reduce the second and fourth components to 0 and the third component to 2. Note that an invariant has not been found which distinguishes these normalizations.

6.1.2 G_3 on V_2^*

(30.8): This space is non-simple G.

6.1.3 G_3 on V_3

The normalizations provided by Petrov have not been independently verified for this section.

(31.5): The abstract Lie algebra (defined by the Killing vector fields) of (31.5) is also found at (31.27), (31.28), and (31.29). Note that (31.5) is non-degenerate on the orbits whereas (31.27), (31.28), and (31.29) have null orbit type (see (31.27), (31.28), and (31.29) below).

(31.6): The abstract Lie algebra of (31.6) is also found at (31.30), (31.31) $\epsilon = 0$, and (31.31) $\epsilon = 1$. Note that (31.6) is non-degenerate on the orbits whereas (31.30), (31.31) $\epsilon = 0$, and (31.31) $\epsilon = 1$ have null orbit type (see (31.30), (31.31) $\epsilon = 0$, and (31.31) $\epsilon = 1$ below).

(31.7): The abstract Lie algebra of (31.7) is also found at (31.32), (31.33), (31.34) $\epsilon = 0$, (31.34) $\epsilon = 1$, and (31.35). Note that (31.7) is non-degenerate on the orbits whereas (31.32), (31.33), (31.34) $\epsilon = 0$, (31.34) $\epsilon = 1$, and (31.35) have null orbit type (see (31.32), (31.33), (31.34) $\epsilon = 0$, (31.34) $\epsilon = 1$, and (31.35) below).

(31.8): Errors were found in the metric. In component g_{33}, we should see e^{2x^1} instead of e^{2x^2} and in component g_{32} we should see ${x^1}^2$ instead of ${x^1}^1$. The abstract Lie algebra of (31.8) is also found at (31.37) $\epsilon = 1, q = 1$, (31.39) $\epsilon = 1, q = 1$, and (31.40). Note that (31.8) is non-degenerate on the orbits whereas (31.37) $\epsilon = 1, q = 1$, (31.39) $\epsilon = 1, q = 1$, and (31.40) have null orbit type (see (31.37) $\epsilon = 1, q = 1$, (31.39) $\epsilon = 1, q = 1$, and (31.40) below).

(31.9): The abstract Lie algebra of (31.9) is also found at (31.37) $\epsilon = 0, q = 1$, (31.39) $\epsilon = 0, q = 1$, and (31.41). Note that (31.9) is non-degenerate on the orbits whereas (31.37) $\epsilon = 0, q = 1$, (31.39) $\epsilon = 0, q = 1$, and (31.41) have null orbit type (see (31.37) $\epsilon = 0, q = 1$, (31.39) $\epsilon = 0, q = 1$, and (31.41) below).

(31.10): The abstract Lie algebra of (31.10) is also found at (31.37) $\epsilon = 0, q \neq 1$, (31.39) $\epsilon = 0, q \neq 1$, and (31.42). Note that (31.10) is non-degenerate on the orbits whereas (31.37) $\epsilon = 0, q \neq 1$, (31.39) $\epsilon = 0, q \neq 1$, and (31.42) have null orbit type (see (31.37) $\epsilon = 0, q \neq 1$, (31.39) $\epsilon = 0, q \neq 1$, and (31.42) below).

(31.11): The abstract Lie algebra of (31.11) is also found at (31.43), and (31.44). Note that (31.11) is non-degenerate on the orbits whereas (31.43), and (31.44) have null orbit type (see (31.43), and (31.44) below).

(31.14): The abstract Lie algebra of (31.14) is also found at (31.45), (31.46), and (31.47). Note that (31.14) is non-degenerate on the orbits whereas (31.45), (31.46), and (31.47) have null orbit type (see (31.45), (31.46), and (31.47) below).

(31.15): The abstract Lie algebra of (31.15) is also found at (31.48). Note that (31.5) is non-degenerate on the orbits whereas (31.48) has null orbit type (see (31.48) below).

6.1.4 G_3 on V_3^*

The normalizations provided by Petrov have not been independently verified for this section.

(31.27): We have a diffeomorphism from (31.27), $\{x^i\}$ to (31.28), $\{y^i\}$ given by

$$\phi(x) = (y^1 = x^3, y^2 = x^1 + \sigma(x^4)x^3, y^3 = x^2 + x^4x^3, y^4 = x^4),$$

where σ is an arbitrary function of x^4. Note that these are abelian Lie algebras of Killing vectors. Note that (31.27) admits a special operator whereas (31.28) does not.

(31.28): See (31.27). We have a diffeomorphism from $(31.28), \{y^i\}$ to $(31.29), \{w^i\}$ given by

$$\psi(y) = (w^1 = y^3 - y^4 y^1, w^2 = y^2 + y^4 y^3 - (\sigma(y^4) + {y^4}^2) y^1, w^3 = y^1, w^4 = y^4),$$

where σ is an arbitrary function of x^4. The abstract Lie algebras are identical. An error was found in this metric. In component g_{33} we should see a_{33} and not a_{23}.

(31.29): See (31.28). Note that (31.29) does not admit a special operator and it is thus unclear what invariant or condition distinguishes this metric from (31.28).

(31.30): We have a diffeomorphism from $(31.30), \{x^i\}$ to $(31, 31)$ $\epsilon = 1, \{y^i\}$ given by

$$\phi(x) = (y^1 = -x^3, y^2 = x^1 + x^4 x^3 + x^4, y^3 = x^2 + x^4, y^4 = x^4).$$

The abstract Lie algebras are identical. Note (31.30) admits a special operator while (31.31) does not.

(31.31): See (31,30). We also have a diffeomorphism from (31.31) $\epsilon = 0, \{x^i\}$ to (31.31) $\epsilon = 1, \{y^i\}$ given by

$$\psi(y) = (y^1 = x^1, y^2 = x^2 + \frac{1}{2} x^4 {x^1}^2 - x^4 x^1, y^3 = x^3 - x^4 x^1 + x^4, y^4 = x^4),$$

the abstract Lie algebras being identical.

(31.32): We have a diffeomorphism from $(31.32), \{x^i\}$ to $(31.33), \{y^i\}$ given by

$$\phi_1(x) = (y^1 = x^2, y^2 = x^1, y^3 = x^3, y^4 = x^4),$$

the abstract Lie algebras being identical. Note that (31.32) admits a special operator as does (31.33). It is unclear what invariants might distinguish these metrics, if any.

(31.33): We have a diffeomorphism from $(31.33), \{y^i\}$ to (31.34) $\epsilon = 1, \{w^i\}$ given by

$$\phi_2(y) = (w^1 = y^3, w^2 = y^2 + y^4 y^3, w^3 = y^1 + y^4 e^{y^3}, w^4 = y^4),$$

the abstract Lie algebras being identical. See (31.32). Note (31.33) admits a special operator while (31.34) does not.

(31.34): We have a diffeomorphism from (31.34) $\epsilon = 1, \{w^i\}$ to (31.35), $\{z^i\}$

$$\phi_3(w) = (z^1 = w^2 - w^4 w^1 + w^4,\\ z^2 = w^3 + w^4 \mathrm{e}^{w^1} w^2 + (-{w^4}^2 w^1 + w^4)\mathrm{e}^{w^1}, z^3 = w^4 \mathrm{e}^{w^1}, z^4 = w^4).$$

with a change of basis needed for (31.35), given by $\{X_2, X_1, X_3\}$, so as to get the structure equations of the abstract Lie algebra identical to those of (31.34) $\epsilon = 1$. Neither (31.34) nor (31.35) admit a special operator and it is unclear what invariants might distinguish these metrics. It should be noted that we have a diffeomorphism from (31.34) $\epsilon = 1, \{x^i\}$ to (31.34) $\epsilon = 0, \{y^i\}$ given by

$$\phi_4(x) = (y^1 = x^1, y^2 = x^2 - x^4 x^1, y^3 = x^3 + \mathrm{e}^{x^1}, y^4 = x^4),$$

with the abstract Lie algebras already identical. See (31.33).

(31.35): There are errors in this metric. In component g_{32} it should read

$$(a_{23} - a_{22} x^1) {x^3}^{-2}$$

and in component g_{42} it should read $a_{24} {x^3}^{-1}$, where $a_{ij} = a_{ij}(x^4)$. See (31.34).

(31.37) $\epsilon = 0, q = 1$: We have a diffeomorphism from (31.37) $\epsilon = 0, q = 1, \{x^i\}$ to (31.39) $\epsilon = 0, q = 1, \{y^i\}$ given by

$$\phi(x) = (y^1 = x^3, y^2 = x^1 + \mathrm{e}^{x^3}, y^3 = x^2, y^4 = x^4),$$

noting the abstract Lie algebras are identical. Note (31.37) admits a special operator while (31.39) does not.

<u>(31.37) $\epsilon = 1, q = 1$</u>: We have a diffeomorphism from (31.37) $\epsilon = 1, q = 1, \{x^i\}$ to (31.39) $\epsilon = 0, q = 1, \{y^i\}$ given by

$$\phi(x) = (y^1 = x^3, y^2 = x^1 + \mathrm{e}^{x^3}, y^3 = x^2, y^4 = x^4),$$

noting the abstract Lie algebras are identical. Note (31.37) admits a special operator while (31.39) does not.

<u>(31.37) $\epsilon = 0, q \neq 1$</u>: We have a diffeomorphism from (31.37) $\epsilon = 0, q \neq 1, \{x^i\}$ to (31.39) $\epsilon = 0, q \neq 1, \{y^i\}$ given by

$$\phi_1(x) = (y^1 = x^3, y^2 = x^1 + \mathrm{e}^{x^3}, y^3 = x^2, y^4 = x^4),$$

noting the abstract Lie algebras are identical. Note (31.37) admits a special operator while (31.39) does not.

<u>(31.39) $\epsilon = 0, q = 1$</u>: See (31.37) $\epsilon = 0, q = 1$.

<u>(31.39) $\epsilon = 1, q = 1$</u>: See (31.37) $\epsilon = 1, q = 1$.

<u>(31.39) $\epsilon = 0, q \neq 1$</u>: We have a diffeomorphism from (31.39) $\epsilon = 0, q \neq 1, \{y^i\}$ to (31.42), $\{w^i\}$, given by

$$\phi_2(y) = (w^1 = y^3, w^2 = y^2 + \mathrm{e}^{(1-q)y^1} y^3, w^3 = \mathrm{e}^{(1-q)y^1}, w^4 = y^4),$$

noting the abstract Lie algebras are identical. See (31.37) $\epsilon = 0, q \neq 1$. Neither (31.39) nor (31.42) admit a special operator and it is unclear what invariants might distinguish these metrics.

<u>(31.42)</u>: There is an error in the metric. In component g_{33}, it should read a_{22} instead of $-a_{22}$, where $a_{22} = a_{22}(x^4)$. See (31.39) $\epsilon = 0, q \neq 1$.

<u>(31.43)</u>: There is an error in the metric. Note that instead of Petrov's g_{34}, we should have $-g_{34}$. Also, we have a diffeomorphism from (31.43), $\{x^i\}$ to (31.44), $\{y^i\}$ given by

$$\phi(x) = \Big(y^1 = x^3, y^2 = x^2 + \left(\frac{1}{2}q - \frac{1}{2}\tanh\left(\frac{1}{2}x^1\sqrt{-4+q^2}\right)\sqrt{-4+q^2}\right)x^3,$$
$$y^3 = \frac{1}{2}q - \frac{1}{2}\tanh\left(\frac{1}{2}x^1\sqrt{-4+q^2}\right)\sqrt{-4+q^2}, y^4 = x^4\Big),$$

noting the abstract Lie algebras are identical. Neither (31.43) nor (31.44) admit a special operator and it is unclear what invariants might distinguish these metrics.

<u>(31.44)</u>: See (31.43).

<u>(31.45)</u>: Note that the abstract Lie algebras of (31.45), (31.46), and (31.47) are identical. However, no diffeomorphism has been found between any of the three vector field systems nor is it clear what invariants may distinguish the general metrics. Note that (31.45), (31.46), and (31.47) do not admit special operators.

6.1.5 G_4 on V_3

<u>(32.3)</u>: This Lie algebra-subalgebra pair is identical to that of (32.18). Note that (32.3) is on V_3 while (32.18) is on $\overset{*}{V}_3$.

<u>(32.4)</u>: This Lie algebra-subalgebra pair is identical to that of (32.20). Note that (32.4) is on V_3 while (32.20) is on $\overset{*}{V}_3$.

<u>(32.5)</u>: Let X be the generators in (32.5) and Y the generators in (32.21). Then $Y_1 = X_2$, $Y_2 = X_1$, $Y_3 = X_3$, and $Y_4 = X_4$ defines a Lie algebra isomorphism between the Lie algebra-subalgebra pairs. Note that (32.5) is on V_3 while (32.21) is on $\overset{*}{V}_3$.

<u>(32.6)</u>: Let X be the generators in (32.6) and Y the generators in (32.22). Then $Y_1 = X_1$, $Y_2 = X_2$, $Y_3 = X_4$, and $Y_4 = X_3$ defines a Lie algebra isomorphism between the Lie algebra-subalgebra pairs. Note that (32.6) is on V_3 while (32.22) is on $\overset{*}{V}_3$.

<u>(32.7)</u>: Let X be the generators in (32.7) and Y the generators in (32.23) $e_2 = -1$. Then $Y_1 = X_3$, $Y_2 = -X_2$, $Y_3 = -X_1$, and $Y_4 = X_4$ defines a Lie algebra isomorphism between the Lie algebra-subalgebra pairs. Now let W be the generators in (32.24), then

$W_1 = X_3$, $W_2 = -X_2$, $W_3 = -X_1$, and $W_4 = X_4$ defines a Lie algebra isomorphism between the Lie algebra-subalgebra pairs. Note that (32.7) is on V_3 while (32.23) $e_2 = -1$ is on $\overset{*}{V}_3$.

There is an error in the metric. We believe Petrov intended $e_3 = 1$ in component g_{33}. Otherwise the vector fields do not preserve the metric. In addition, the vector fields contain an error. The correct form of X_1 should be

$$X_1 = \cos x^3 p_2 - (\coth x^2 \sin x^3 - 1)p_3,$$

where in Petrov's notation $p_i = \partial_{x^i}$, as otherwise they do not form a Lie algebra. Petrov also claims that if one changes the trigonometric functions in the vector fields (32.7) to the corresponding hyperbolic functions, one obtains another set of Killing vectors. This isn't true as in that case the vector fields do not form a Lie algebra.

(32.8): There is a small typo in the metric in component g_{33}. Petrov has x_2^2 when it should be ${x^2}^2$.

(32.11): In Petrov, X_4 should be $X_4 = x^2 p_3 - \epsilon x^3 p_2$, where $\epsilon = \pm 1$ is a parameter in the given metric. This greatly clarifies the situation of determining which set of vector fields belong with which metric. Also, there is a small typo in the metric. In component g_{44}, Petrov has dx^4 which should be ${dx^4}^2$.

6.1.6 G_4 on V_3^*

(32.18): The general invariant metric of the Killing vectors admits additional isometries for an isometry dimension of six and is non-simple G. See (32.3) and Table 6.7.

(32.20): There is an error in the metric. We should see dx^2dx^4 and not dx^2dx^3. However, the general invariant metric of the Killing vectors admits additional isometries for an isometry dimension of six and is non-simple G. See (32.4) and Table 6.7.

(32.21): The given metric has non-Lorentzian signature which is found by gauge fixing the general invariant metric of the given vector fields under the assumption that the first component $s_1 = 0$ (see $[4, 3, 12]$ in Appendix A.2). However, by assuming $s_1 \neq 0$, a second normalization is admitted that is spacelike or timelike on the orbits and is equivalent to (32.5).

(32.22): See (32.6).

(32.23) $e_2 = 1$: The given metric has non-Lorentzian signature. Petrov's metric can be found by gauge fixing the general invariant metric of the given vector fields under the assumption that the first component $s_1 = 0$ (see $[4, 3, 8]$ in Appendix A.2). However, by assuming $s_1 \neq 0$, a normalization not given by Petrov is admitted that is spacelike or timelike on the orbits.

(32.24) $e_2 = 1$: There is an error in the Killing vectors. The correct form for X_3 should be $X_3 = -\mathrm{e}^{x^3} p_1 + \left(x^{2^2} + \mathrm{e}^{2x^3}\right) p_2 + 2x^2 p_3$.

The given vector fields admit a general invariant metric g which can be normalized by assuming the first and second components satisfy $s_2(x4) = -s_1(x4)$ (see $[4, 3, 9]$ in Appendix A.2). This normalization is given by Petrov and is null on the orbits. However, it has non-Lorentzian signature. On the other hand, for $s_2(x4) \neq -s_1(x4)$, a normalization is admitted that is not given by Petrov which is spacelike or timelike on the orbits.

(32.24) $e_2 = -1$: The vector fields admit a general invariant metric g which can be normalized by assuming the first and second components satisfy $s_2(x4) = s_1(x4)$ (see $[4, 3, 2]$ in Appendix A.2). This gives null orbit type and does indeed have Lorentzian signature (see (32.24) $e_2 = 1$ above). However, a second normalization is admitted by assuming $s_2(x4) \neq s_1(x4)$. This normalization is not given by Petrov and is spacelike or timelike on the orbits.

(32.26): Not a simple G space.

(32.27): The general invariant metric admits two additional isometries and becomes non-simple G. See (32.11).

6.1.7 G_4 on V_4

The normalizations provided by Petrov have not been independently verified for this section.

<u>(32.38)</u>: There is a diffeomorphism from (32.38), $\{x^i\}$ to (32.39), $\{y^i\}$ given by

$$\phi(x) = \Big(y^1 = x^1 - \alpha x^4, y^2 = x^2 + x^4 e^{\alpha x^4 - x^1}, y^3 = x^3, y^4 = x^4\Big),$$

where α is a constant. The abstract Lie algebras are identical. It is unclear if these metrics are equivalent.

<u>(32.39)</u>: See (32.38).

<u>(32.43), $\epsilon = 0$</u>: There is a diffeomorphism from (32.43), $\epsilon = 0, \{x^i\}$ to (32.44), $\epsilon = 0, \{y^i\}$, given by

$$\phi(x) = \Big(y^1 = x^2/k^2 - x^3/k + x^1 + (-x^4/k + 1/k^2 - 1/k)e^{kx^4}, \\ y^2 = x^2 + e^{kx^4}, y^3 = x^3 + (x^4+1)e^{kx^4}, y^4 = x^4\Big).$$

The abstract Lie algebras of $(32, 43, 1)$ and $(32, 44, 1)$ are identical. It remains to be resolved whether or not the metrics are equivalent.

<u>(32.43), $\epsilon = 1$</u>: There is a diffeomorphism from (32.43), $\epsilon = 1, \{x^i\}$ to (32.44), $\epsilon = 1, \{y^i\}$, given by

$$\phi(x) = \Big(y^1 = \frac{x^2}{(-1+k)^2} - \frac{x^3}{(-1+k)} + x^1 - \frac{e^{kx^4 - x^4}(kx^4 + k - 2 - x^4)e^{x^4}}{(-1+k)^2}, \\ y^2 = x^2 + e^{kx^4}, y^3 = x^3 + (x^4+1)e^{kx^4}, y^4 = x^4\Big).$$

The abstract Lie algebras of $(32, 43, 1)$ and $(32, 44, 1)$ are identical. It remains to be resolved whether or not the metrics are equivalent.

<u>(32.44), $\epsilon = 0$</u>: See (32.43), $\epsilon = 0$.

<u>(32.44), $\epsilon = 1$</u>: See (32.43), $\epsilon = 1$.

<u>(32.46)</u>: We have a diffeomorphism from (32.46) $\epsilon = 1, \{x^i\}$ to (32.46) $\epsilon = 1, \{y^i\}$, given by

$$\phi(x) = (x^1 = y^1, x^2 = y^2, x^3 = y^3, x^4 = -y^4).$$

The abstract Lie algebras are identical.

6.1.8 G_5 on V_4

(33.5): There is an error in the metric. Petrov has $\lambda = \alpha(k+1) + \beta(\nu - k) \neq 0$ which should read $\lambda = \alpha(k+1) + \beta(1-k)$.

The given metric was over normalized. The first component of the general invariant metric must be an arbitrary constant as this parameterizes the Ricci scalar (see $[5, 4, -6]$ in Appendix A.2). The general invariant metric of the given vector fields has vanishing Riemann invariants, the exception being the Ricci scalar.

If X represents the Killing vectors for $(33.5), \epsilon = 0$, then its Lie algebra-subalgebra pair is isomorphic to $(33.8), \epsilon = 0$ by the change of basis

$$\{X_1, X_2, X_3, \frac{-(k-1)(\alpha+\beta)X_4}{\beta(1-k)+\alpha(k+1)} + \frac{2kX_5}{\beta(1-k)+\alpha(k+1)}, X_5\}$$

and we have a diffeomorphism from $(33.5), \epsilon = 0, \{x^i\}$ to $(33.8), \epsilon = 0, \{w^i\}$ given by

$$\phi_1(x) = \Big(w^1 = x^1, w^2 = x^2 + \mathrm{e}^{(-\beta k+\beta+\alpha k+\alpha)x^4}\mathrm{e}^{-x^1 k}, w^3 = x^3,$$
$$w^4 = \frac{-kx^1}{\alpha+\beta} + \frac{(x^4(-k+1)\beta + x^4(k+1)\alpha)}{\alpha+\beta}\Big),$$

where α, β, k are constants which satisfy $\alpha(k+1) + \beta(1-k) \neq 0$.

A diffeomorphism from $(33.5), \epsilon = 0, \{x^i\}$ to $(33.5), \epsilon = -1, \{y^i\}$ is given by

$$\phi_2(x) = \Big(y^1 = x^1, y^2 = x^2 + \Big[\frac{2}{-\beta k+\beta+\alpha k+\alpha} + \mathrm{e}^{(-\beta k+\beta+\alpha k+\alpha)x^4}\Big]\mathrm{e}^{-x^1 k},$$
$$y^3 = x^3, y^4 = x^4\Big),$$

noting the Lie algebra-subalgebra pairs are identical. Also, we have a diffeomorphism from $(33.5), \epsilon = -1, \{y^i\}$ to $(33.5), \epsilon = 0, \{w^i\}$, given by

$$\phi_3(y) = \Big(w^1 = y^1, w^2 = y^2 + \Big(\frac{-1}{-\beta k+\beta+\alpha k+\alpha} + \mathrm{e}^{(-\beta k+\beta+\alpha k+\alpha)y^4}\Big)\mathrm{e}^{-y^1 k},$$
$$w^3 = y^3, w^4 = y^4\Big),$$

noting that again the Lie algebra-subalgebra pairs are identical.

For $\epsilon = \pm 1$, (33.5) is Petrov type "I" and for $\epsilon = 0$ is type "III". But (33.8) is Petrov type "N".

(33.6): The given metric was over normalized. The fourth component of the general invariant metric must be an arbitrary constant as this parameterizes the Ricci scalar (see $[5, 4, -3]$ in Appendix A.2). The general invariant metric of the given vector fields has vanishing Riemann invariants, the exception being the Ricci scalar.

We have a diffeomorphism from $(33.6), \epsilon = 1, \{x^i\}$ to $(33.6), \epsilon = -1, \{y^i\}$, given by

$$\phi_1(x) = (y^1 = x^1, y^2 = x^2 + (-1 + \mathrm{e}^{-2x^4})\mathrm{e}^{2x^1}, y^3 = x^3, y^4 = x^4),$$

and from $(33.6), \epsilon = 1, \{x^i\}$ to $(33.6), \epsilon = 0, \{w^i\}$, given by

$$\phi_2(x) = (w^1 = x^1, w^2 = x^2 + \left(\frac{-1}{2} + \mathrm{e}^{-2x^4}\right)\mathrm{e}^{2x^1}, w^3 = x^3, w^4 = x^4),$$

noting that the three Lie algebra-subalgebra pairs are identical.

We also have a diffeomorphism from $(33.6), \epsilon = 0, \{x^i\}$ to $(33.29), \epsilon = 0,_k = 1/2l, \{y^i\}$, given by

$$\phi_3(x) = (y^1 = x^2 + \mathrm{e}^{-2x^4}\mathrm{e}^{2x^1}, y^2 = x^1, y^3 = -x^3, y^4 = \frac{2x^1}{l} - \frac{2x^4}{l}),$$

where if Y is the set of generators for $(33.29), \epsilon = 0,_k = 1/2l$, then it is isomorphic to $(33.6), \epsilon = 0$ by the change of basis $\{-Y_3, -Y_2, Y_4, Y_1 + \frac{2Y_5}{l}, Y_1\}$, noting that Petrov gives $2k = l + \epsilon$ and thus in this case $l \neq 0$.

Lastly note that the metric of (33.6) is Petrov type "N" while that of $(33.29), \epsilon = 0,_k = 1/2l$ is type "I".

(33.7): The given metric was over normalized. The first component of the general invariant metric must be an arbitrary constant as this parameterizes the Ricci scalar (see

$[5, 4, -5]$ in Appendix A.2). The general invariant metric of the given vector fields has vanishing Riemann invariants, the exception being the Ricci scalar.

We have a diffeomorphism from $(33.7), \epsilon = 1, \{x^i\}$ to $(33.7), \epsilon = -1, \{y^i\}$, given by

$$\phi_1(x) = (y^1 = x^1, y^2 = x^2 + (1 + \mathrm{e}^{2x^4})\mathrm{e}^{-qx^1}, y^3 = x^3, y^4 = x^4)$$

and then from $(33.7), \epsilon = 1, \{x^i\}$ to $(33.7), \epsilon = 0, \{w^i\}$

$$\phi_2(x) = (w^1 = x^1, w^2 = x^2 + (\frac{1}{2} + \mathrm{e}^{2x^4})\mathrm{e}^{-qx^1}, w^3 = x^3, w^4 = x^4),$$

noting the Lie algebra-subalgebra pairs are identical.

It remains to be resolved whether or not the metrics are equivalent.

<u>(33.8)</u>: See (33.5). There is a small typo in the metric. In component g_{11}, we should have $\mathrm{e}^{-\lambda x^4}$, and not $\mathrm{e}^{+\lambda x^4}$, noting $\lambda = \alpha + \beta$.

The given metric was over normalized. The first component of the general invariant metric must be an arbitrary constant as this parameterizes the Ricci scalar (see $[5, 4, -6]$ in Appendix A.2). The general invariant metric of the given vector fields has vanishing Riemann invariants, the exception being the Ricci scalar.

We have a diffeomorphism from $(33.8), \epsilon = 1, \{x^i\}$ to $(33.8), \epsilon = -1, \{y^i\}$, given by

$$\phi_1(x) = (y^1 = x^1, y^2 = x^2 + \frac{2x^1}{\alpha+\beta} + \frac{2}{\alpha+\beta} + \mathrm{e}^{(\alpha+\beta)x^4}, y^3 = x^3, y^4 = x^4)$$

and then from $(33.8), \epsilon = 1, \{x^i\}$ to $(33.8), \epsilon = 0, \{w^i\}$, given by

$$\phi_2(x) = (w^1 = x^1, w^2 = x^2 + \frac{x^1}{\alpha+\beta} + \frac{1}{\alpha+\beta} + \mathrm{e}^{(\alpha+\beta)x^4}, w^3 = x^3, w^4 = x^4),$$

noting that if X, Y are the bases for $(33.8), \epsilon = 1$ and $(33.8), \epsilon = -1$ respectively, then the changes of basis

$$\{X_1, X_2, X_3, \frac{-2X_1}{\alpha+\beta} + X_4, \frac{2\beta X_1}{\alpha+\beta} + X_5\}$$

and

$$\{Y_1, Y_2, Y_3, \frac{2Y_1}{\alpha+\beta} + Y_4, \frac{-2\beta Y_1}{\alpha+\beta} + Y_5\}$$

show that these are identical Lie algebra-subalgebra pairs.

It remains to be resolved whether or not the metrics are equivalent.

(33.12): Petrov's given metric was under normalized. We were able to reduce the first and third components of the general invariant metric to ϵ_1 and ϵ_2 with $\epsilon_1, \epsilon_2 = \pm 1$ (see $[5, 4, -2]$ in Appendix A.2). Also, the general invariant metric has vanishing Riemann invariants, the exception being the Ricci scalar.

We have a diffeomorphism from $(33.12), \{x^i\}$ to $(33.13), \{y^i\}$, given by

$$\phi_1(x) = (y^1 = -x^1, y^2 = -x^2 - \frac{1}{2}{x^3}^2 \mathrm{e}{-x^1}, y^3 = -x^3 \mathrm{e}^{-x^1}, y^4 = -x^4)$$

and from $(33.12), \{x^i\}$ to $(33.29), \epsilon = 1, k = \frac{l}{2} + \frac{1}{2}, \{w^i\}$, given by

$$\begin{aligned}\phi_2(x) = \Big(w^1 = x^2 + \frac{1}{2}{x^3}^2 \mathrm{e}^{-x^1} + \Big(k_2 \frac{a}{(1+a)(2+a)} + k_2 \frac{a^2}{(1+a)(2+a)} + \\ + \frac{1}{2} \frac{a k_{44} \mathrm{e}^{\frac{(2+a)x^4}{a} + x^4}}{k_{12}(1+a)}\Big) \mathrm{e}^{-\frac{(2+a)x^1}{a}}, w^2 = -\mathrm{e}^{-x^1}, w^3 = -x^3 \mathrm{e}^{-x^1}, \\ w^4 = -x^1 + x^4\Big),\end{aligned}$$

noting that if X, Y, and W are the generators of $(33.12), \{x^i\}$, $(33.13), \{y^i\}$, and $(33.29), \epsilon = 1, k = \frac{l}{2} + \frac{1}{2}, \{w^i\}$ respectively, then the following changes of basis show the respective Lie algebra-subalgebra pairs are identical to one another:

$$\{-X_1, X_3, X_5, X_2, X_4\},$$

with $k = 2/a + 1$, and

$$\{Y_1, Y_2, -Y_5, Y_3, \frac{k-1}{k+1} Y_4\},$$

with $k = -2/a - 1$, and

$$\{W_3, W_2, W_1, W_4, \frac{2}{l-1} W_5\},$$

with $l = 2/a + 1$.

It remains to be resolved whether or not the metrics are equivalent. Lastly, note that if $k = 0$, then (33.12) and (33.13) have symmetry classification $[6, 4, -1]$, a non-reductive case of G_6 on V_4.

(33.13): See (33.12). Petrov's given metric was under normalized. We were able to reduce the first and third components of the general invariant metric to ϵ_1 and ϵ_2 with $\epsilon_1, \epsilon_2 = \pm 1$ (see $[5, 4, -2]$ in Appendix A.2). Also, the general invariant metric has vanishing Riemann invariants, the exception being the Ricci scalar.

(33.14): The given metric was over normalized. The fourth component of the general invariant metric must be an arbitrary constant as this parameterizes the Ricci scalar (see $[5, 4, 10]$ in Appendix A.2). The general invariant metric of the given vector fields has vanishing Riemann invariants, the exception being the Ricci scalar. See (33.18) for additional information regarding this Petrov entry.

$(33, 16)$: The given metric was under normalized. The second and fourth components of the general invariant metric can be reduced to $\epsilon_1, \epsilon_2 = \pm 1$ respectively (see $[5, 4, 11]$ in Appendix A.2).

(33.17) $\epsilon = -1$: There is an error in the metric. It should be $4dx^1dx^4$ and not $2dx^1dx^4$. Also, there is a missed normalization in Petrov. In the general invariant metric, by assuming the first and second components satisfy $s_1 = s_2$, one arrives at the normalization of Petrov. However, considering the case $s_1 \neq s_2$, there is another possible normalization not given by Petrov (see the example in the introduction to this chapter and $[5, 4, 1]$ in Appendix A.2). The missed normalization carried within it the Gödel metric, which cannot be in Petrov's given normalization.

(33.17) $\epsilon = 1$: The given metric has non-Lorentzian signature. However, there is a missed normalization in Petrov. In the general invariant metric, by assuming the first and second components satisfy $s_1 = s_2$, one arrives at the normalization of Petrov, a metric with non-Lorentzian signature. Considering the case $s_1 \neq s_2$, there is another possible normalization not given by Petrov (see $[5, 4, 7]$ in Appendix A.2).

(33, 18): There is an error in the metric. The g_{11} component should read

$$-2k_{12}x^4 \mathrm{e}^{-2\epsilon x^4} + k_{11}\mathrm{e}^{-2\epsilon x^4},$$

and note term $dx^1 dx^1$ is missing entirely. Also, the parameter ϵ should take values $+1$ or -1, and not 0 or 1. If $\epsilon = 0$ as stated, the vector fields do form a Lie algebra but the general invariant metric of those vector fields has maximal symmetry.

Also, the given metric was under normalized. The first component of the general invariant metric can be reduced to $\epsilon = \pm 1$ (see $[5, 4, 10]$ in Appendix A.2).

The Lorentzian pairs given by (33.14) and (33.18) (as corrected) are found in $[5, 4, 10]$ in Appendix A.2. Note for either $\epsilon = \pm 1$ that $[5, 4, 10]$ is Petrov type "N". Also, (33.14) (generically) and (33.18) $\epsilon = 1$ are type "N". It remains to be resolved if the metrics of (33.14) and (33.18) $\epsilon = 1$ are equivalent. However, (33.18) $\epsilon = -1$ is conformally flat and thus Petrov type "O". Note that $[5, 4, 10]$, (33.14), (33.18) $\epsilon = 1$, and (33.18) $\epsilon = -1$ all have vanishing Riemann invariants, except the Ricci scalar for $[5, 4, 10]$, (33.14), and (33.18) $\epsilon = 1$.

(33.19): We have a diffeomorphism from (33.19), $\{x^i\}$ to (33.20), $\{y^i\}$ given by

$$\phi(x) = (y^1 = x^3, y^2 = x^2 - \frac{\pi}{2}, y^3 = \frac{\pi}{2} - x^1, y^4 = x^4),$$

noting that the Lie algebra-subalgebra pairs are identical.

The given metric was under normalized as the fourth component of the general invariant metric can be put to $\epsilon = \pm 1$ (see $[5, 4, 2]$ in Appendix A.2). Another normalization of the general invariant metric is possible with the second and fourth components 0 and the third component to 1 which is the metric Petrov gives in (33.20) (see $[5, 4, 2]$ in Appendix A.2). Note that (33.19) is Petrov type "D" and (33.20) is Petrov type "N".

(33.20): See (33.19).

(33.21): For the case $c \neq 0$, the given metric was under normalized as the first component of the general invariant metric can be put to $\epsilon = \pm 1$, reducing to two free constants instead three (see $[5,4,8]$ in Appendix A.2). For $c = 0$, one can reduce the first component to one and the fourth component to $\epsilon = \pm 1$ (see $[5,4,6]$ in Appendix A.2).

(33.22): The given metric was under normalized as the second component of the general invariant metric can be moved to $\epsilon = \pm 1$ (see $[5,4,4]$ in Appendix A.2).

(33.23): The given metric was under normalized as the second and fourth components of the general invariant metric can be moved to $\epsilon_1, \epsilon_2 = \pm 1$ respectively (see $[5,4,3]$ in Appendix A.2).

(33.24): We have a diffeomorphism from (33.24), $\{x^i\}$ to (33.27), $\{y^i\}$ given by

$$\phi(x) = (y^1 = -x^1 + 2\alpha x^4 + x^4, y^2 = x^3, y^3 = x^2, y^4 = x^4),$$

where α is a constant in (33.24), and (33.27) needing a simple change of basis first, namely $\{Y_2, Y_1, Y_3, Y_4, Y_5\}$, making the structure equations of the two Lie algebra-subalgebra pairs identical. However, the general invariant metric admits additional isometries giving symmetry classification $[7,4,4]$. These systems were claimed as G_5 on V_4.

(33.25): We have a diffeomorphism from (33.25), $\{x^i\}$ to (33.26), $\{y^i\}$, given by

$$\phi(x) = (y^1 = -2x^1 + 3\alpha x^4, y^2 = x^2, y^3 = x^3, y^4 = x^4),$$

where α is a constant in both $(33, 25)$ and $(33, 26)$, noting the structure equations of the two Lie algebra-subalgebra pairs are identical. However, the general invariant metric admits additional isometries giving symmetry classification $[7,4,2]$. These systems were claimed as G_5 on V_4.

(33.26): The metric had a small typo in that it was missing the $g_{44} = k_{44}$ component. See (33.25).

(33.27): The metric had a small typo in that it was missing the $g_{44} = k_{44}$ component. See (33.24).

(33.28): Within the fifth vector field X_5, there is a small typo. It should read $+p_4$ and not $= p_4$. The given metric was under normalized as the first component of the general invariant metric can be put to $\epsilon = \pm 1$ and the third component to 1 (see $[5, 4, 9]$ in Appendix A.2).

We have a diffeomorphism from $(33.28), \epsilon = 0, \{x^i\}$ to $(33.28), \epsilon = 1, \{y^i\}$

$$\phi(x) = (y^1 = x^1 + \mathrm{e}^{lx^4}, y^2 = x^2, y^3 = x^3, y^4 = x^4),$$

where one would substitute $k+1$ for k in $(33.28), \epsilon = 0$ and change its basis by the following $\{X_1, X_2, X_3, X_4, X_4 + X_5\}$.

$\underline{(33.29), \epsilon = 1}$: The given metric was under normalized as the first and second components of the general invariant metric can be put to $\epsilon_1, \epsilon_2 = \pm 1$ respectively (see $[5, 4, -2]$ in Appendix A.2). Note $l \neq 0$ in (33.29) as this gives a flat spacetime. See (33.12).

$\underline{(33.29), \epsilon = 0}$: The given metric was under normalized as the first and second components of the general invariant metric can be put to $\epsilon_1, \epsilon_2 = \pm 1$ respectively (see $[5, 4, -3]$ in Appendix A.2) . See (33.6).

$\underline{(33.30)}$: There are errors in the metric. In g_{11} we should see e^{-2x^4} instead of e^{-x^4}. In g_{12} and g_{33} we should see e^{-4x^4} instead of e^{-2x^4}. The given metric was under normalized as the first and second components of the general invariant metric can be put to $\epsilon_1, \epsilon_2 = \pm 1$ respectively (see $[5, 4, -4]$ in Appendix A.2).

$\underline{(33.31)}$: The given metric was under normalized as the first and third components of the general invariant metric can be put to $\epsilon_1, \epsilon_2 = \pm 1$ respectively (see $[5, 4, 5]$ in Appendix A.2).

$\underline{(33.36)}$: There is an error in the metric. We should have $g_{11} = -k_{11}\mathrm{e}^{x^4}$ and not $k_{11}\mathrm{e}^{x^4}$. Also, there is an error in the Killing vectors. In X_5, both exponential functions should have positive exponents and no minus signs. However, the general invariant metric has non-Lorentzian signature and thus this entry is excluded from comparison with our work.

$\underline{(33.37)}$: The general invariant metric has non-Lorentzian signature and thus this entry is excluded from comparison with our work.

<u>(33.38)</u>: We have an error in the metric. We should see $g_{22} = \frac{k_{22}}{x^{2^2}}$ and not $g_{22} = k_{22}\mathrm{e}^{\lambda x^4+2x^1}$. Also, we should see in the metric $2\lambda\varepsilon k_{11}\mathrm{e}^{-x^4}dx^1dx^3$ and not $\lambda\varepsilon k_{11}\mathrm{e}^{-x^4}dx^1dx^3$, as the metric is symmetric.

Also, the change of basis $\{X_2, X_1+X_3, -\lambda X_1, \lambda X_1+X_4, X_5\}$ on $(33.38), \epsilon=1$ shows it to be identical to $(33.39), \epsilon=1$, as abstract Lie algebras. However, the general invariant metric admits additional isometries giving symmetry classification $[7,4,4]$.

<u>(33.39)</u>: See (33.38). The general invariant metric admits two additional isometries giving symmetry classification $[7,4,4]$. See Table 6.9.

6.1.9 G_6 on V_3

No vector fields were given in Petrov for this case. The Killing vector fields were computed from the metrics given.

<u>(33.40) A, B, C</u>: These metrics constitute Petrov's entries for G_6 on V_3. The Killing vectors of (C) do not define a simple G space. See Table 6.10.

6.1.10 G_6 on V_4

No vector fields were given in Petrov for this case. The Killing vector fields were computed from the metrics given.

<u>(33.44), (33.45)</u>: Petrov claims these to be G_6 acting on V_4. However, they are both G_6 on V_3. See Table 6.10.

<u>(33.48), (33.49)</u>: The cases (33.48), $e_1=1, e_2=-1$ and (33.49), $e_1=1, e_2=-1$ admit equivalent Lorentzian Lie algebra-subalgebra pairs. However, it is not clear if there is an invariant which distinguishes the metrics. The cases (33.48), $e_1=-1, e_2=1$ and (33.49), $e_1=-1, e_2=1$ are unique and have symmetry classification $[6,4,4]$ and $[6,4,5]$ respectively.

<u>(33.50)</u>: The cases (33.50) and (33.51), $e_1=-1, e_2=1$ admit equivalent Lorentzian Lie algebra-subalgebra pairs (see $[6,4,2]$ in Appendix A.2). However, it is not clear if there is an invariant which distinguishes the metrics.

(33.51): See (33.50). Also, the cases (33.51), $e_1 = 1, e_2 = -1$ and (33.52) admit equivalent Lorentzian Lie algebra-subalgebra pairs (see $[6, 4, 2]$ in Appendix A.2). However, it is not clear if there is an invariant which distinguishes the metrics.

(33.52): See (33.51).

6.1.11 G_7 on V_4

No vector fields were given in Petrov for this case. The Killing vector fields were computed from the metrics given.

(33.42): This metric gives the Lorentzian Lie algebra-subalgebra pair $[7, 4, 5]$. However, as a parameterized Lie algebra it contains five parameters whereas $[7, 4, 5]$ of this classification has three.

This concludes the commentary on Chapter 5 of Petrov's *Einstein Spaces* [2].

6.2 Symmetry classification tables

This section gives tables containing the symmetry classification *according to the classification of this dissertation* for the entries in Petrov [2]. The cases for G_3 on V_2 are in Table 6.5; G_3 on V_3 are in Table 6.6; G_4 on V_3 are in Table 6.7; G_4 on V_4 are in Table 6.8; G_5 on V_4 are in Table 6.9; G_6 on V_3, G_6 on V_4, and G_7 on V_4 are in Table 6.10.

TABLE 6.5: Symmetry Classification of G_3 on V_2 in Petrov

Entry	Database	Isotropy Type	Classification
(30,1)	[30, 1, 0]	F12	[3, 2, 1]
(30.2)	[30, 2, 0]	F13	[3, 2, 4]
(30.3)	[30, 3, 0]	F12	[3, 2, 2]
(30.4)	[30, 4, 0]	F13	[3, 2, 5]
(30.5)	[30, 5, 0]	F13	[3, 2, 5]
(30.6)	[30, 6, 0]	F12	[3, 2, 3]
(30.8)	[30, 8, 0]		Non-simple G

Table 6.6: Symmetry Classification of G_3 on V_3 in Petrov.

Entry	Database	Classification
(31.5)	[31, 5, 0]	[3, 3, 2]
(31.6)	[31, 6, 0]	[3, 3, 3]
(31.7)	[31, 7, 0]	[3, 3, 1]
(31.8)	[31, 8, 0]	[3, 3, 6]
(31.9)	[31, 9, 0]	[3, 3, 5]
(31.10)	[31, 10, 0]	[3, 3, 4]
(31.11)	[31, 11, 0]	[3, 3, 7]
(31.14)	[31, 14, 0]	[3, 3, 8]
(31.15)	[31, 15, 0]	[3, 3, 9]
(31.27)	[31, 27, 0]	[3, 3, 2]
(31.28)	[31, 28, 0]	[3, 3, 2]
(31.29)	[31, 29, 0]	[3, 3, 2]
(31.30)	[31, 30, 0]	[3, 3, 3]
(31.31), $\epsilon = 0$	[31, 30, 0]	[3, 3, 3]
(31.31), $\epsilon = 1$	[31, 30, 1]	[3, 3, 3]
(31.32)	[31, 32, 0]	[3, 3, 1]
(31.33)	[31, 33, 0]	[3, 3, 1]
(31.34), $\epsilon = 0$	[31, 34, 0]	[3, 3, 1]
(31.34), $\epsilon = 1$	[31, 34, 1]	[3, 3, 1]
(31.35)	[31, 35, 0]	[3, 3, 1]
(31.37), $\epsilon = 1$, $q = 1$	[31, 37, 0]	[3, 3, 6]
(31.37), $\epsilon = 0$, $q = 1$	[31, 37, 1]	[3, 3, 5]
(31.37), $\epsilon = 0$, $q \neq 1$	[31, 37, 2]	[3, 3, 4]
(31.39), $\epsilon = 1$, $q = 1$	[31, 39, 0]	[3, 3, 6]
(31.39), $\epsilon = 0$, $q = 1$	[31, 39, 1]	[3, 3, 5]
(31.39), $\epsilon = 0$, $q \neq 1$	[31, 39, 2]	[3, 3, 4]
(31.40)	[31, 40, 0]	[3, 3, 6]
(31.41)	[31, 41, 0]	[3, 3, 5]
(31.42)	[31, 42, 0]	[3, 3, 4]
(31.43)	[31, 43, 0]	[3, 3, 7]
(31.44)	[31, 44, 0]	[3, 3, 7]
(31.45)	[31, 45, 0]	[3, 3, 8]
(31.46)	[31, 46, 0]	[3, 3, 8]
(31.47)	[31, 47, 0]	[3, 3, 8]
(31.48)	[31, 48, 0]	[3, 3, 9]

Table 6.7: Symmetry Classification of G_4 on V_3 in Petrov. $*$ indicates additional symmetries are admitted.

Entry	Database	Isotropy Type	Classification
(32.3)	[32, 3, 0]	F13	[4, 3, 10]
(32.4)	[32, 4, 0]	F12	[4, 3, 5]
(32.5)	[32, 5, 0]	F13	[4, 3, 12]
(32.6)	[32, 6, 0]	F12	[4, 3, 7]
(32.7)	[32, 7, 0]	F12	[4, 3, 1]
(32.8)	[32, 8, 0]	F14	[4, 3, 20]
(32.9)	[32, 9, 0]	F12	[4, 3, 3]
(32.10)	[32, 10, 0]	F12	[4, 3, 4]
(32.11), $\epsilon = -1$	[32, 11, 0]	F13	[4, 3, 11]
(32.11), $\epsilon = 1$	[32, 11, 1]	F12	[4, 3, 6]
(32.12)	[32, 12, 0]	F14	[4, 3, 19]
(32.14), $c \neq 0$, $c \neq 1$	[32, 14, 0]	F14	[4, 3, 15]
(32.14), $c = 1$	[32, 14, 1]	F14	[4, 3, 13]
(32.14), $c = 0$	[32, 14, 2]	F14	[4, 3, 18]
(32.15)	[32, 15, 0]	F14	[4, 3, 14]
(32.16), $q \neq 0$	[32, 16, 0]	F14	[4, 3, 16]
(32.16), $q = 0$	[32, 16, 1]	F14	[4, 3, 17]
(32.18)$*$	[32, 18, 0]	Non-simple G	NA
(32.20)$*$	[32, 20, 0]	Non-simple G	NA
(32.21), non-Lorentzian metric	[32, 21, 0]	F13	[4, 3, 12]
(32.22)	[32, 22, 0]	F12	[4, 3, 7]
(32.23), $e_2 = -1$	[32, 23, 0]	F12	[4, 3, 1]
(32.23), $e_2 = 1$, non-Lorentzian metric	[32, 23, 1]	F13	[4, 3, 8]
(32.24), $e_2 = -1$	[32, 24, 0]	F12	[4, 3, 2]
(32.24), $e_2 = 1$, non-Lorentzian metric	[32, 24, 1]	F13	[4, 3, 9]
(32.25), $\epsilon = 0$	[32, 25, 0]	F12	[4, 3, 3]
(32.25), $\epsilon = 1$	[32, 25, 1]	F12	[4, 3, 4]
(32.26)	[32, 26, 0]	Non-simple G	NA
(32.27), $e = -1*$	[32, 27, 0]	F4	[6, 3, 5]
(32.27), $e = 1*$	[32, 27, 1]	F4	[6, 3, 5]

Table 6.8: Symmetry Classification of G_4 on V_4 in Petrov

Entry	Database	Classification
(32.34), $c \neq 0$, $c \neq 2$	[32, 34, 0]	[4, 4, 16]
(32.34), $c = 0$	[32, 34, 1]	[4, 4, 18]
(32.34), $c = 2$	[32, 34, 1]	[4, 4, 17]
(32.35)	[32, 35, 0]	[4, 4, 19]
(32.36)	[32, 36, 0]	[4, 4, 20]
(32.37), $q \neq 0$	[32, 37, 0]	[4, 4, 21]
(32.37), $q = 0$	[32, 37, 1]	[4, 4, 22]
(32.38)	[32, 38, 0]	[4, 4, 8]
(32.39)	[32, 39, 0]	[4, 4, 8]
(32.40)	[32, 40, 0]	[4, 4, 23]
(32.41), $\epsilon = 0$	[32, 41, 0]	[4, 4, 5]
(32.41), $\epsilon = 1$	[32, 41, 1]	[4, 4, 10]
(32.41), $\epsilon = 0$, $k = -1$, $l = -1$	[32, 41, 2]	[4, 4, 6]
(32.41), $\epsilon = 1$, $k = -1$, $l = 1$	[32, 41, 3]	[4, 4, 11]
(32.42)	[32, 42, 0]	[4, 4, 15]
(32.42), $l = 0$	[32, 42, 1]	[4, 4, 3]
(32.43), $\epsilon = 0$	[32, 43, 0]	[4, 4, 4]
(32.43), $\epsilon = 1$	[32, 43, 1]	[4, 4, 12]
(32.43), $\epsilon = 1$, $k = 0$	[32, 43, 2]	[4, 4, 7]
(32.43), $\epsilon = 1$, $k = 1$	[32, 43, 3]	[4, 4, 13]
(32.44), $\epsilon = 0$	[32, 44, 0]	[4, 4, 4]
(32.44), $\epsilon = 1$	[32, 44, 1]	[4, 4, 12]
(32.45), $\epsilon = 0$	[32, 45, 0]	[4, 4, 9]
(32.45), $\epsilon = 1$	[32, 45, 1]	[4, 4, 14]
(32.46), $\epsilon = -1$	[32, 46, 0]	[4, 4, 2]
(32.46), $\epsilon = 1$	[32, 46, 1]	[4, 4, 2]
(32.47)	[32, 47, 0]	[4, 4, 1]

TABLE 6.9: Symmetry Classification of G_5 on V_4 in Petrov. $*$ indicates additional symmetries are admitted.

Entry	Database	Isotropy Type	Classification
(33.1)	[33, 1, 0]	Non-simple G	NA
(33.5), $\epsilon = -1$	[33, 5, 0]	F14	[5, 4, -6]
(33.5), $\epsilon = 0$	[33, 5, 1]	F14	[5, 4, -6]
(33.5), $\epsilon = 1$	[33, 5, 2]	F14	[5, 4, -6]
(33.6), $\epsilon = -1$	[33, 6, 0]	F14	[5, 4, -3]
(33.6), $\epsilon = 0$	[33, 6, 1]	F14	[5, 4, -3]
(33.6), $\epsilon = 1$	[33, 6, 2]	F14	[5, 4, -3]
(33.7), $\epsilon = -1$	[33, 7, 0]	F14	[5, 4, -5]
(33.7), $\epsilon = 0$	[33, 7, 1]	F14	[5, 4, -5]
(33.7), $\epsilon = 1$	[33, 7, 2]	F14	[5, 4, -5]
(33.8), $\epsilon = -1$	[33, 8, 0]	F14	[5, 4, -6]
(33.8), $\epsilon = 0$	[33, 8, 1]	F14	[5, 4, -6]
(33.8), $\epsilon = 1$	[33, 8, 2]	F14	[5, 4, -6]
(33.12)	[33, 12, 0]	F14	[5, 4, -2]
(33.13)	[33, 13, 0]	F14	[5, 4, -2]
(33.14)	[33, 14, 0]	F14	[5, 4, 10]
(33.16)	[33, 16, 0]	F14	[5, 4, 11]
(33.17), $\epsilon = -1$	[33, 17, 0]	F12	[5, 4, 1]
(33.17), $\epsilon = 1$	[33, 17, 0]	F13	[5, 4, 7]
(33.18), $\epsilon = -1$	[33, 18, 0]	F14	[5, 4, 10], $\epsilon = -1$
(33.18), $\epsilon = 1$	[33, 18, 1]	F14	[5, 4, 10], $\epsilon = 1$
(33.19)	[33, 19, 0]	F12	[5, 4, 2]
(33.20)	[33, 20, 0]	F12	[5, 4, 2]
(33.21)	[33, 21, 0]	F13	[5, 4, 8]
(33.21), $c = 0$	[33, 21, 1]	F13	[5, 4, 6]
(33.22)	[33, 22, 0]	F12	[5, 4, 4]
(33.23)	[33, 23, 0]	F12	[5, 4, 3]
(33.24)$*$	[33, 24, 0]	F4	[7, 4, 4]
(33.25)$*$	[33, 25, 0]	F3	[7, 4, 2]
(33.26)$*$	[33, 26, 0]	F4	[7, 4, 2]
(33.27)$*$	[33, 27, 0]	F4	[7, 4, 4]
(33.28), $\epsilon = 0$	[33, 28, 0]	F13	[5, 4, 9]
(33.28), $\epsilon = 1$	[33, 28, 1]	F13	[5, 4, 9]
(33.29), $\epsilon = 0$	[33, 29, 0]	F14	[5, 4, -3]
(33.29), $\epsilon = 1$	[33, 29, 1]	F14	[5, 4, -2]
(33.30)	[33, 30, 0]	F14	[5, 4, -4]
(33.31)	[33, 31, 0]	F12	[5, 4, 5]
(33.36)	[33, 36, 0]	Non-Lorentzian signature	NA
(33.37)	[33, 37, 0]	Non-Lorentzian signature	NA
(33.38), $\epsilon = -1$ $*$	[33, 38, 0]	F13	[7, 4, 4]
(33.38), $\epsilon = 1$ $*$	[33, 38, 1]	F13	[7, 4, 4]
(33.39), $\epsilon = -1$ $*$	[33, 39, 0]	F13	[7, 4, 4]
(33.39), $\epsilon = 1$ $*$	[33, 39, 1]	F13	[7, 4, 4]

TABLE 6.10: Symmetry Classification of G_6 and G_7 on V_3 and V_4 in Petrov. † Petrov claimed these to be G_6 on V_4. †† [33, 41, 0] has the 3-subspace of positive constant curvature, while [33, 41, 1] has the 3-subspace of negative constant curvature.

Entry	Database	Classification	Isotropy Type
(33.40), (A), $e_1 = -1$, $e_2 = -1$, $e_3 = -1$, $e_4 = 1$	[33, 40, 0]	[6, 3, 5]	F4
(33.40), (A), $e_1 = 1$, $e_2 = -1$, $e_3 = -1$, $e_4 = -1$	[33, 40, 1]	[6, 3, 2]	F3
(33.40), (B), $e_1 = -1$, $e_2 = -1$, $e_3 = -1$, $e_4 = 1$	[33, 40, 2]	[6, 3, 3]	F3
(33.40), (B), $e_1 = 1$, $e_2 = 1$, $e_3 = 1$, $e_4 = -1$	[33, 40, 3]	[6, 3, 6]	F4
(33.40), (C)	[33, 40, 4]	NA	Non-simple G
(33.41)	[33, 41, 0]††	[7, 4, 1]	F3
(33.41)	[33, 41, 1]	[7, 4, 2]	F3
(33.42)	[33, 42, 0]	[7, 4, 5]	F6
(33.44) †	[33, 44, 0]	[6, 3, 2]	F3
(33.45) †	[33, 45, 0]	[6, 3, 6]	F4
(33.48), $e_1 = -1$, $e_2 = 1$	[33, 48, 0]	[6, 4, 4]	F9
(33.48), $e_1 = 1$, $e_2 = -1$	[33, 48, 1]	[6, 4, 3]	F9
(33.49), $e_1 = -1$, $e_2 = 1$	[33, 49, 0]	[6, 4, 5]	F9
(33.49), $e_1 = 1$, $e_2 = -1$	[33, 49, 1]	[6, 4, 3]	F9
(33.50), $e_1 = -1$, $e_2 = 1$	[33, 50, 0]	[6, 4, 1]	F9
(33.50), $e_1 = 1$, $e_2 = -1$	[33, 50, 1]	[6, 4, 1]	F9
(33.51), $e_1 = -1$, $e_2 = 1$	[33, 51, 0]	[6, 4, 1]	F9
(33.51), $e_1 = 1$, $e_2 = -1$	[33, 51, 1]	[6, 4, 2]	F9
(33.52), $e_1 = -1$, $e_2 = 1$	[33, 52, 0]	[6, 4, 2]	F9
(33.52), $e_1 = 1$, $e_2 = -1$	[33, 52, 1]	[6, 4, 2]	F9
(33.54)	[33, 54, 0]	[6, 4, 6]	F10

CHAPTER 7

KOMRAKOV'S CLASSIFICATION OF EINSTEIN-MAXWELL SPACETIMES

The classifier software *HomogeneousSpaceClassifier* and database of Chapter 4 are put to use to give the classification of Lorentzian pairs given in the paper by B. Komrakov, [7]. The focus in [7] is on four-dimensional homogeneous spaces G/H with dim(G) ≥ 5 and dim(H) ≥ 1. The purpose for this lies in the paper's goals of classifying Einstein-Maxwell spacetimes on such spaces. The reader is referred to [7] for more information.

The subalgebras of $\mathfrak{so}(4)$, $\mathfrak{so}(2,2)$, and $\mathfrak{so}(3,1)$ have been classified and Komrakov lists these subalgebras in Theorem 1 of [7]. In Table 7.1 below, the labeling of these subalgebras is given. Also listed is the number of entries of pseudo-Riemannian pairs for which the isotropy acts as the particular subalgebra. We also give the beginning page number for which the pairs are summarized in [7].

In this dissertation, we are concerned with Lorentzian pairs, which are special cases of pseudo-Riemannian pairs. Thus in Table 7.2 the information from Table 7.1 relevant to our work is given, namely the Lorentzian Lie algebra-subalgebra pairs of Table 7.1.

All pairs in Table 7.2 have been classified against the classification of Lorentzian pairs given in Chapter 3 using the software of Chapter 4. On the following pages we will give tables summarizing the classification of Lorentzian pairs found in Komrakov [7]. It's important to recognize that the curvature tensor of G-invariant metrics on G/H can be computed from the structure constants of $\mathfrak{g}$ and a given ad($\mathfrak{h}$)-invariant inner product. See Coquereaux [19], page 74, for further details. Consequently, one can check directly whether or not the ad($\mathfrak{h}$)-invariant inner product is giving a flat metric or a metric of constant curvature, or more generally one may simply compute the isometry dimension (see Section 2.6). Thus the isometry dimension was computed for every Komrakov entry by computing the general ad($\mathfrak{h}$)-invariant inner product. Whenever the isometry dimension exceeded the dimension of the Lie algebra pair, *meaning additional symmetries were admitted*, a note in the table is made. An indication of "Flat" simply means the general invariant metric admits ten isometries and has vanishing curvature. "Constant curvature" means the general invariant metric admits ten isometries and has constant curvature.

Table 7.1: Pseudo-Riemannian Pairs in Komrakov.

Subalgebra	# of Entries	Lie Algebra	Page
1.1^1	10	$\mathfrak{so}(3,1)$, $\mathfrak{so}(2,2)$	42
1.1^2	12	$\mathfrak{so}(3,1)$, $\mathfrak{so}(2,2)$, $\mathfrak{so}(4)$	45
1.1^3	1	$\mathfrak{so}(3,1)$	49
1.1^4	1	$\mathfrak{so}(3,1)$	50
1.1^5	1	$\mathfrak{so}(2,2)$	50
1.1^6	1	$\mathfrak{so}(2,2)$	50
1.2^1	1	$\mathfrak{so}(2,2)$	51
1.2^2	1	$\mathfrak{so}(2,2)$	51
1.3^1	32	$\mathfrak{so}(2,2)$	52
1.4^1	26	$\mathfrak{so}(3,1)$, $\mathfrak{so}(2,2)$	65
2.1^1	3	$\mathfrak{so}(2,2)$	74
2.1^2	6	$\mathfrak{so}(3,1)$	75
2.1^3	6	$\mathfrak{so}(4)$, $\mathfrak{so}(2,2)$	77
2.1^4	2	$\mathfrak{so}(2,2)$	79
2.2^1	7	$\mathfrak{so}(2,2)$	80
2.2^2	4	$\mathfrak{so}(2,2)$	83
2.2^3	1	$\mathfrak{so}(2,2)$	85
2.3^1	1	$\mathfrak{so}(2,2)$	85
2.4^1	3	$\mathfrak{so}(3,1)$, $\mathfrak{so}(2,2)$	86
2.5^1	14	$\mathfrak{so}(2,2)$	87
2.5^2	7	$\mathfrak{so}(3,1)$	93
3.1^1	1	$\mathfrak{so}(2,2)$	96
3.1^2	1	$\mathfrak{so}(2,2)$	97
3.2^1	4	$\mathfrak{so}(2,2)$	97
3.2^2	2	$\mathfrak{so}(3,1)$	99
3.3^1	4	$\mathfrak{so}(2,2)$	100
3.3^2	4	$\mathfrak{so}(3,1)$	102
3.4^1	1	$\mathfrak{so}(2,2)$	104
3.4^2	1	$\mathfrak{so}(4)$	105
3.5^1	4	$\mathfrak{so}(3,1)$, $\mathfrak{so}(2,2)$	105
3.5^2	4	$\mathfrak{so}(3,1)$, $\mathfrak{so}(4)$	107
4.1^1	1	$\mathfrak{so}(2,2)$	109
4.1^2	1	$\mathfrak{so}(3,1)$	109
4.2^1	2	$\mathfrak{so}(2,2)$	110
4.2^2	3	$\mathfrak{so}(4)$	111
4.2^3	2	$\mathfrak{so}(2,2)$	113
4.3^1	2	$\mathfrak{so}(2,2)$	115
5.1^1	1	$\mathfrak{so}(2,2)$	117
6.1^1	2	$\mathfrak{so}(2,2)$	117
6.1^2	3	$\mathfrak{so}(4)$	118
6.1^3	3	$\mathfrak{so}(3,1)$	120

As an example, take the pair 1.1^1, number 5, on page 43 of [7]. The Lie table is given by the following:

Table 7.2: Lorentzian Pairs in Komrakov.

Subalgebra	# of Entries	Lie Algebra	Page
1.1^1	10	$\mathfrak{so}(3,1)$, $\mathfrak{so}(2,2)$	42
1.1^2	12	$\mathfrak{so}(3,1)$, $\mathfrak{so}(2,2)$, $\mathfrak{so}(4)$	45
1.1^3	1	$\mathfrak{so}(3,1)$	49
1.1^4	1	$\mathfrak{so}(3,1)$	50
1.4^1	26	$\mathfrak{so}(3,1)$, $\mathfrak{so}(2,2)$	65
2.1^2	6	$\mathfrak{so}(3,1)$	75
2.4^1	3	$\mathfrak{so}(3,1)$, $\mathfrak{so}(2,2)$	86
2.5^2	7	$\mathfrak{so}(3,1)$	93
3.2^2	2	$\mathfrak{so}(3,1)$	99
3.3^2	4	$\mathfrak{so}(3,1)$	102
3.5^1	4	$\mathfrak{so}(3,1)$, $\mathfrak{so}(2,2)$	105
3.5^2	4	$\mathfrak{so}(3,1)$, $\mathfrak{so}(4)$	107
4.1^2	1	$\mathfrak{so}(3,1)$	109
6.1^3	3	$\mathfrak{so}(3,1)$	120

	e_1	u_1	u_2	u_3	u_4
e_1	.	u_1	.	$-u_3$	.
u_1		.	.	e_1	.
u_2			.	.	u_2
u_3				.	.
u_4					.

Note that $\mathfrak{g} = \text{span}\{e_1, u_1, \ldots, u_4\}$ and the isotropy subalgebra is $\mathfrak{h} = \text{span}\{e_1\}$. Let $\mathfrak{m} = \{u_1, \ldots, u_4\}$, then $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ as a vector space direct sum with $\mathfrak{m}$ a reductive complement. We may initialize the Lie algebra in Maple:

$$> LD := LieAlgebraData(['[e1,u1] = u1',' [e1,u3] = -u3',' [u1,u3] = e1',' [u2,u4] = u2',$$
$$'[u3,u4] = 0'],' [e1,u1,u2,u3,u4]', alg) :$$
$$> DGsetup(LD) :$$

However, we have the following relations between the basis vectors in Maple, $\{e_1, e_2, \ldots, e_5\}$, and the basis vectors in the notation of Komrakov, $\{e_1, u_1, \ldots, u_4\}$. These are $e_1 = e_1,\ e_2 = u_1,\ e_3 = u_2,\ e_4 = u_3$, and $e_5 = u_4$. Here is the multiplication table given in terms of Maple's vectors:

	e_1	e_2	e_3	e_4	e_5
e_1	.	e_2	.	$-e_4$	.
e_2		.	.	e_1	.
e_3			.	.	e_3
e_4				.	.
e_5					.

We can use the Maple command *DGEnvironment*[*GSpace*] to initialize a simple G space.

$$> DGEnvironment[GSpace]([e2, e3, e4, e5],\ [e1],\ G,\ vectorlabels\ =\ [X],$$
$$formlabels\ =\ [sigma]);$$

Note that $\mathfrak{m} = \{e_2, e_3, e_4, e_5\}$ and $\mathfrak{h} = \{e_1\}$. We use the options $vectorlabels\ =\ [X]$ and $formlabels\ =\ [sigma]$ to designate the labels for the vectors and forms according to our liking. Once the space G is initialized in Maple, X_1 through X_4 represent the complement and X_5 the isotropy and σ_i the dual basis. We use the command *GenerateSymmetricTensors* to generate all rank-2 covariant symmetric tensors on G and the command *InvariantGeometricObjectFields* to create the most general ad($\mathfrak{h}$)-invariant inner product g on $\mathfrak{g}/\mathfrak{h}$:

$$> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4], 2) :$$
$$> g := InvariantGeometricObjectFields([X5], S);$$
$$g := \frac{\_C1}{2}\sigma 1 \otimes \sigma 3 + \_C2\sigma 2 \otimes \sigma 2 + \frac{\_C3}{2}\sigma 2 \otimes \sigma 4 + \frac{\_C1}{2}\sigma 3 \otimes \sigma 1 + \frac{\_C3}{2}\sigma 4 \otimes \sigma 2$$
$$+ \_C4\sigma 4 \otimes \sigma 4$$

The $\_C1$ through $\_C4$ are arbitrary constants. We are now able to run *IsometryAlgebraData* and initialize its output. The output will be suppressed though the multiplication table of the isometry algebra is displayed below. With the option *output* = [“*All*”], *IsometryAlgebraData* returns the dimension of the isometry algebra, named *dim*, the Lie algebra structure equations for the isometry algebra, named LDx, and the components of the isotropy subalgebra in the basis of LDx, named $isocomp$.

$> dim,\ LDx,\ isocomp := IsometryAlgebraData(g,\ [\,],\ output\ =\ [\text{“}All\text{”}]):$

$> DGsetup(LDx):$

$\quad Liealgebra: \_isomalg1$

We see the dimension of the isometry algebra is 6:

$> dim;$

$\quad 6$

The components of the isotropy are the following:

$> isocomp;$

$\quad [[0,0,0,0,1,0],[0,0,0,0,0,1]]$

Therefore the isotropy is given by the span of the fifth and sixth vectors. Here is the multiplication table for the isometry algebra as given by LDx (not displayed above):

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	$\frac{\_C1}{2}e_5$	.	$-\frac{2}{\_C1}e_5$	.
e_2		.	.	$-\_C2e_6$	.	$\frac{2\_C3}{4\_C2\_C4-\_C3^2}e_2-\frac{4\_C2}{4\_C2\_C4-\_C3^2}e_4$
e_3			.	.	$\frac{2}{\_C1}e_3$	.
e_4				.	.	$\frac{4\_C4}{4\_C2\_C4-\_C3^2}e_2-\frac{2\_C3}{4\_C2\_C4-\_C3^2}e_4$
e_5					.	.
e_6						.

We can then call the command *HomogeneousSpaceClassifier* (see Chapter 4) to classify the Lorentzian pair. Recall that we need to supply a list containing the name of the Lie algebra (in this case the name is $\_isomalg1$) and a list defining the basis for the isotropy subalgebra. Observe that the output below is $[6,4,2]$, which is what is shown in Table 7.3. Also, as indicated in Table 7.3, there is an asterisk associated to $[6, 4, 2]$. The asterisk indicates this case admitted an additional symmetry since the isometry algebra dimension is 6 and the Lie algebra of pair 1.1^1, number 5, has dimension 5. It is in this fashion that the tables found in this chapter were populated and in which it was determined for which cases additional symmetries were admitted by the general G-invariant metrics associated to the Lorentzian pairs given in Komrakov [7].

$> ChangeFrame(\_isomalg1):$

$> HomogeneousSpaceClassifier([\_isomalg1, [e5, e6]]);$

$$[[6, 4, 2]]$$

7.1 Tables

In Table 7.3 is the classification for Komrakov's 1.1^1; in Table 7.4 is the classification for Komrakov's 1.1^2; in Table 7.5 is the classification for Komrakov's 1.1^3; in Table 7.6 is the classification for Komrakov's 1.1^4; in Table 7.7 is the classification for Komrakov's 1.4^1; in Table 7.8 is the classification for Komrakov's 2.1^2; in Table 7.9 is the classification for Komrakov's 2.4^1; in Table 7.10 is the classification for Komrakov's 2.5^2; in Table 7.11 is the classification for Komrakov's 3.2^2; in Table 7.12 is the classification for Komrakov's 3.3^2; in Table 7.13 is the classification for Komrakov's 3.5^1; in Table 7.14 is the classification for Komrakov's 3.5^2; in Table 7.15 is the classification for Komrakov's 4.1^2; and in Table 7.16 is the classification for Komrakov's 6.1^3.

TABLE 7.3: Classification of Lorentzian pairs 1.1^1 in Komrakov. $*$ indicates additional symmetries admitted by invariant metric.

Pair #	Classification	Isotropy Type
1	[5, 4, 8]	F13
2	[5, 4, 9]	F13
3	[5, 4, 7]	F13
4	[5, 4, 6]	F13
5∗	[6, 4, 2]	F9
6∗	[6, 4, 5]	F9
7∗	[6, 4, 3]	F9

TABLE 7.4: Classification of Lorentzian pairs 1.1^2 in Komrakov. $*$ indicates additional symmetries admitted by invariant metric.

Pair #	Classification	Isotropy Type
1	[5, 4, 4]	F12
2	[5, 4, 5]	F12
3	[5, 4, 2]	F12
4	[5, 4, 1]	F12
5	[5, 4, 3]	F12
6$*$	[6, 4, 1]	F9
7$*$	[6, 4, 2]	F9
8$*$	[6, 4, 3]	F9
9$*$	[6, 4, 4]	F9
10$*$	[6, 4, 5]	F9

TABLE 7.5: Classification of Lorentzian pair 1.1^3 in Komrakov.

Pair #	Classification
1	Flat

TABLE 7.6: Classification of Lorentzian pair 1.1^4 in Komrakov.

Pair #	Classification
1	Flat

TABLE 7.7: Classification of Lorentzian pairs 1.4^1 in Komrakov. $*$ indicates additional symmetries admitted by invariant metric.

Pair #	Classification	Isotropy Type
1	[5, 4, -1]	F14
2	[5, 4, -2]	F14
3	[5, 4, -6]	F14
4	[5, 4, -5]	F14
5	[5, 4, 11]	F14
6	[5, 4, 10], $\epsilon = 1$	F14
7	[5, 4, 10], $\epsilon = -1$	F14
8	Constant Curvature	
9-22, 24, 25, $*$	[6, 4, 6]	F10
23	Flat	
26	Flat	

TABLE 7.8: Classification of Lorentzian pairs 2.1^2 in Komrakov.

Pair #	Classification	Isotropy Type
1	[6, 4, 1]	F9
2	[6, 4, 2]	F9
3	[6, 4, 3]	F9
4	[6, 4, 4]	F9
5	[6, 4, 5]	F9
6	Flat	

TABLE 7.9: Classification of Lorentzian pairs 2.4^1 in Komrakov. $*$ indicates additional symmetries admitted by invariant metric.

Pair #	Classification	Isotropy Type
1$*$	[7, 4, 4]	F4
2	Constant Curvature	
3	Flat	

TABLE 7.10: Classification of Lorentzian pairs 2.5^2 in Komrakov.

Pair #	Classification	Isotropy Type
1	[6, 4, -1]	F10
2-6	[6, 4, 6]	F10
7	Flat	

TABLE 7.11: Classification of Lorentzian pairs 3.2^2 in Komrakov.

Pair #	Classification	Isotropy Type
1	[7, 4, -1]	F7
2	Flat	

TABLE 7.12: Classification of Lorentzian pairs 3.3^2 in Komrakov.

Pair #	Classification	Isotropy Type
1-3	[7, 4, 5]	F6
4	Flat	

Table 7.13: Classification of Lorentzian pairs 3.5^1 in Komrakov.

Pair #	Classification	Isotropy Type
1	Constant Curvature	
2	[7, 4, 4]	F4
3	[7, 4, 3]	F4
4	Flat	

Table 7.14: Classification of Lorentzian pairs 3.5^2 in Komrakov.

Pair #	Classification	Isotropy Type
1	Constant Curvature	
2	[7, 4, 2]	F3
3	[7, 4, 1]	F3
4	Flat	

Table 7.15: Classification of Lorentzian pairs 4.1^2 in Komrakov

Pair #	Classification
1	Flat

Table 7.16: Classification of Lorentzian pairs 6.1^3 in Komrakov.

Pair #	Classification
1-2	Constant Curvature
3	Flat

CHAPTER 8

SYMMETRY CLASSIFICATION OF EXACT SOLUTIONS

In this chapter we present in tables the use of the software developed in Chapters 4 and 5 to give the symmetry classification of solutions to the Einstein equations found in the book "Exact Solutions of Einstein's Field Equations" by Stephani et al. [1].

In Tables 8.1 and 8.2 we give the symmetry classification of exact solutions from Chapter 12 of Stephani. Table 8.3 gives the classification of exact solutions from Chapter 28 of Stephani.

TABLE 8.1: Classification of exact solutions – Stephani, Chapter 12. A listing of the symmetry classification of exact solutions found in Chapter 12 of Stephani.

Solution	Type	Classification
[12, 6, 1]	Pure Radiation	[6, 4, -1]
[12, 7, 1]	Einstein-Maxwell	2-dimensional
[12, 8, 1]	Einstein	[6, 4, 1]
[12, 8, 2]	Generic	[6, 4, 4]
[12, 8, 3]	Generic	[6, 4, 1]
[12, 8, 4]	Generic	[6, 4, 3]
[12, 8, 5]	Generic	[6, 4, 3]
[12, 8, 6]	Generic	[6, 4, 2]
[12, 8, 7]	Generic	[6, 4, 5]
[12, 8, 8]	Einstein	[6, 4, 2]
[12, 9, 1]	Generic	[6, 3, 1]
[12, 9, 2]	Generic	[6, 3, 2]
[12, 9, 3]	Generic	[6, 3, 3]
[12, 12, 1]	Einstein-Maxwell	[6, 4, 6]
[12, 12, 2]	Einstein-Maxwell	[7, 4, 5]
[12, 12, 3]	Einstein-Maxwell	[6, 4, 6]
[12, 12, 4]	Einstein-Maxwell	[7, 4, 5]
[12, 13, 1]	Vacuum	[6, 4, 6]
[12, 14, 1]	Vacuum	[4, 4, 15]
[12, 16, 1]	Einstein-Maxwell	[6, 4, 1]
[12, 18, 1]	Einstein-Maxwell	[6, 4, 1]
[12, 19, 1]	Einstein-Maxwell	[6, 4, 1]
[12, 21, 1]	Einstein-Maxwell	[4, 4, 18]
[12, 23, 1]	Perfect Fluid	[7, 4, 1]
[12, 23, 2]	Perfect Fluid	[7, 4, 2]
[12, 24.1, 1]	Perfect Fluid	[7, 4, 1]
[12, 24.2, 1]	Perfect Fluid	[7, 4, 1]
[12, 24.3, 1]	Perfect Fluid	[7, 4, 1]

Table 8.2: Classification of exact solutions – Stephani, Chapter 12 continued.

Solution	Type	Classification
[12, 26, 1]	Perfect Fluid	[5, 4, 1]
[12, 27, 1]	Perfect Fluid	[4, 4, 1]
[12, 28, 1]	Perfect Fluid	[4, 4, 2]
[12, 29, 1]	Perfect Fluid	[4, 4, 2]
[12, 30, 1]	Perfect Fluid	[4, 4, 10]
[12, 31, 1]	Perfect Fluid	[4, 4, 12]
[12, 32, 1]	Perfect Fluid	[4, 4, 15]
[12, 34, 1]	Einstein	[5, 4, -2]
[12, 35, 1]	Einstein	[4, 4, 10]
[12, 36, 1]	Einstein-Maxwell	[5, 4, -2]
[12, 37, 1]	Pure Radiation	[6, 4, 6]
[12, 37, 2]	Einstein-Maxwell	[6, 4, 6]
[12, 37, 3]	Pure Radiation	[7, 4, 5]
[12, 37, 4]	Einstein-Maxwell	[7, 4, 5]
[12, 37, 5]	Pure Radiation	[7, 4, 5]
[12, 37, 6]	Einstein-Maxwell	[7, 4, 5]
[12, 37, 7]	Einstein-Maxwell	[7, 4, 5]
[12, 37, 8]	Einstein-Maxwell	[7, 4, 5]
[12, 37, 9]	Einstein-Maxwell	[7, 4, 5]
[12, 38, 1]	Pure Radiation	[5, 4, -2]
[12, 38, 2]	Pure Radiation	[5, 4, -4]
[12, 38, 3]	Einstein-Maxwell	[5, 4, -2]
[12, 38, 4]	Pure Radiation	[5, 4, -2]
[12, 38, 5]	Einstein	[5, 4, -2]

TABLE 8.3: Classification of exact solutions – Stephani, Chapter 28. A listing of the symmetry classification of exact solutions found in Stephani et al., Chapter 28.

Solution	Type	Classification
$[28, 16, 1]$	Vacuum	$[3, 3, 4]$
$[28, 21, 1]$	Vacuum	$[4, 3, 3]$
$[28, 21, 2]$	Vacuum	$[4, 3, 3]$
$[28, 21, 3]$	Vacuum	$[4, 3, 3]$
$[28, 21, 4]$	Vacuum	$[4, 3, 6]$
$[28, 21, 5]$	Vacuum	$[4, 3, 6]$
$[28, 21, 6]$	Vacuum	$[4, 3, 1]$
$[28, 21, 7]$	Vacuum	$[4, 3, 1]$
$[28, 43, 1]$	Einstein-Maxwell	$[3, 2, 1]$
$[28, 44, 1]$	Einstein-Maxwell	$[4, 3, 3]$
$[28, 44, 2]$	Einstein-Maxwell	$[4, 3, 3]$
$[28, 44, 3]$	Einstein-Maxwell	$[4, 3, 1]$
$[28, 44, 4]$	Einstein-Maxwell	$[4, 3, 6]$
$[28, 44, 5]$	Einstein-Maxwell	$[4, 3, 1]$
$[28, 44, 6]$	Einstein-Maxwell	$[4, 3, 1]$
$[28, 72, 1]$	Pure Radiation	$[3, 3, 4]$

CHAPTER 9

CONCLUSION

The aims of this dissertation were the following:

i) Give a new and different approach to the classification of spacetimes with symmetry using modern methods and tools such as the Schmidt method (see Schmidt [6]) and computer algebra systems.
ii) Create computer-based databases of the classification for easy access and use for researchers.
iii) Create software to classify any spacetime metric with symmetry against the new database.
iv) Classify spacetimes with symmetry in Stephani [1] using the new software.

These goals find satisfaction in the following summary.

By definition, a spacetime with symmetry admits a Lie algebra of Killing vectors Γ. Locally Γ defines a Lorentzian Lie algebra-subalgebra pair $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{h}$ abstractly giving a subalgebra of $\mathfrak{so}(3,1)$. This fact motivated Chapter 3 which gave a complete classification of Lorentzian Lie algebra-subalgebra pairs, the results of which are found in Appendix A.1.

Chapter 4 discussed the creation of a database of the classification of Lorentzian pairs given in Chapter 3. Included in this database are enough Lie theoretic invariants to distinguish every pair in the classification one from another. Also discussed in Chapter 4 was the software created for this dissertation called *HomogeneousSpaceClassifier*. This takes as input a Lorentzian Lie algebra-subalgebra pair and classifies it against the database of such pairs according to the Lie theoretic invariants. It then returns the database label of the unique Lorentzian pair possessing the corresponding invariants. Note that the database has been tested successfully against itself using this software.

Chapter 4 also provided a discussion on the software *IsomorphismForLiePairs* written for this dissertation which takes as input two Lie algebra-subalgebra pairs and attempts to find a matrix defining an isomorphism between the pairs. We then discussed *SpacetimeSymmetryClassifier*, a program created by simply adapting *HomogeneousSpaceClassifier* to take as input a Killing algebra of vector fields Γ of a four-dimensional Lorentzian metric and a point p at which the isotropy subalgebra is to be computed. As Γ at p defines a Lorentzian pair, this adapted software then classifies the pair and returns the database label corresponding to a unique Lorentzian pair in the database. This gives researchers the ability to classify any simple G spacetime with symmetry with $3 \leq \dim(\mathrm{G}) \leq 7$. The creation of the software *SpacetimeSymmetryClassifier* signaled a shift in our work from a purely algebraic point of view to a geometric one.

In Chapter 5 (with full results in Appendix A.2) we associated to each Lorentzian pair $(\mathfrak{g}, \mathfrak{h})$ a spacetime (M, g). Using mainly algebraic methods, this was done by first constructing from each Lorentzian pair a Lie algebra of vector fields Γ which at a preferred point determines the abstract pair $(\mathfrak{g}, \mathfrak{h})$. Note that at this step Γ would be considered a vector field system on G/H, where G is the Lie group of $\mathfrak{g}$, $H \subset G$ the Lie group of $\mathfrak{h}$, and $\dim(G/H) \leq 4$. Again using algebraic methods, however, we then constructed on a four-dimensional space M a basis $\mathfrak{G}$ of Γ-invariant quadratic forms independent over the ring of Γ-invariant functions. From $\mathfrak{G}$ we then formed the most general invariant metric tensor g and proceeded to normalize or gauge fix the metric. Note that the full isometry algebra of g was verified to be $\mathfrak{g}$. This normalizing process returns an equivalent metric $\tilde{g}$ having Γ as Killing algebra and possibly fewer unknowns in its local coordinate presentation. However, no normalizing was performed for the cases of G_3 on V_3 and G_4 on V_4 (cases of trivial isotropy), this being reserved as a future project.

With the noted exceptions of $[6, 4, 6]$ and $[7, 4, 5]$, all Lorentzian pairs in Appendix A.1 were realized as vector field systems. We also provided invariant quadratic forms and normalized invariant metric tensors. The vector field systems and normalized invariant metric tensors comprise the classification of simple G spacetimes with symmetry of dimension three through seven and can be found in Appendix A.2. The case splitting required by $[6, 4, 6]$ and $[7, 4, 5]$ is a future project.

In Chapter 6 independent verification of Petrov's classification of spacetimes with symmetry given in the book *Einstein Spaces* [2] was discussed and begun. Regarding Petrov's classification, we

i) identified and corrected typos and small errors in Petrov;

ii) identified Petrov entries for which the Killing vector field systems are diffeomorphic and give explicit diffeomorphisms;

iii) identified Petrov entries for which the given metric is not the most general invariant metric, allowing for proper gauge fixing;

iv) identified Petrov entries for which the Killing vector field system for the given metric is larger than that provided by Petrov;

v) identified the non-reductive entries in Petrov;

vi) identified non-simple G Killing vector field systems in Petrov;

vii) gave the symmetry classification of each simple G entry in Petrov using the software from Chapter 4;

viii) identified the reductive Lorentzian pairs from 3 and the non-reductive pairs from Fels [5] which do not appear in Petrov.

We note that to complete an independent verification of Petrov, one must classify the non-simple G spacetimes and compute the normalizations for the cases of trivial isotropy, namely G_3 on V_3 and G_4 on V_4.

In Chapter 7, the classifier software and databases of chapters 4 and 5 were used to give the classification of Lorentzian pairs found in the paper by B. Komrakov, [7]. The focus in [7] is on four-dimensional homogeneous spaces G/H with dim(G) ≥ 5 and dim(H) ≥ 1 for the purposes of classifying Einstein-Maxwell spacetimes on such spaces. The Komrakov classification of Lorentzian pairs and that of this dissertation are in complete agreement. However, as seen in Chapter 7, there are many non-maximal Lorentzian pairs included in the Komrakov classification.

Similarly in Chapter 8 we present in tables the symmetry classification of solutions to the Einstein equations found in the book "Exact Solutions of Einstein's Field Equations" by Stephani [1]. This is done specifically for Chapters 12 and 28 of [1] using the software created in Chapter 4. As a future project, many more metrics with symmetry from Stephani will be classified. It is the expectation of this author that new solutions to the Einstein equations will be discovered through this process.

BIBLIOGRAPHY

[1] H. Stephani, D. Kramer, M. Maccallum, C. Hoenselaers, and E. Herlt, *Exact solutions of Einstein field equations.* Cambridge, UK: Cambridge University Press, 2003.

[2] A. Z. Petrov, *Einstein spaces.* Pergamon Press Ltd., 1969.

[3] J. Hicks, "Algebraic properties of Killing vectors for Lorentz metrics in four dimensions," Master's thesis, Utah State University, All Graduate Plan B and other Reports. Paper 102., 2011.

[4] A. Bowers, "An algebraic construction of Lorentz homogeneous spaces of low dimension," *Journal of Lie Theory*, vol. 22, no. 3, pp. 887–906, 2012.

[5] M. E. Fels and A. G. Renner, "Non-reductive homogeneous pseudo-Riemannian manifolds of dimension four," *Canadian Journal of Mathematics*, vol. 58, no. 2, p. 282?311, 2006.

[6] B. Schmidt, "Homogeneous Riemannian spaces and Lie algebras of Killing fields," *General Relativity and Gravitation*, vol. 2, no. 2, pp. 105–120, 1971.

[7] B. B. Komrakov, "Einstein-Maxwell equation on four-dimensional homogeneous spaces," *Lobachevskii Journal of Mathematics*, vol. 8, no. 0, pp. 33–165, 2001.

[8] S. Kobayashi, *Transformation groups in differential geometry.* Berlin Heidelberg New York: Springer-Verlag, 1972.

[9] W. M. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry.* San Diego, California: Academic Press, 2003.

[10] J. Rozum, "Classification of five-dimensional lie algebras with one-dimensional subalgebras acting as subalgebras of the Lorentz algebra," Master's thesis, Utah State University, All Graduate Theses and Dissertations. Paper 4633., 2015.

[11] L. Šnobl and P. Winternitz, *Classification and identification of Lie algebras*, vol. 33. American Mathematical Soc., 2014.

[12] W. Fulton and J. Harris, *Representation Theory: A First Course*. New York: Springer-Verlag, 1991.

[13] K. Erdmann and M. J. Wildon, *Introduction to Lie Algebras*. London: Springer-Verlag, 2006.

[14] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*, vol. 94. Springer Science & Business Media, 2013.

[15] W. Greub, *Linear algebra*. New York: Springer-Verlag, 1967.

[16] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. 2. New York: John Wiley and Sons, Inc., 1963.

[17] L. Eisenhart, "Continuous groups of transformations," *Princeton, London*, 1934.

[18] A. Ashtekar and A. Magnon-Ashtekar, "A technique for analyzing the structure of isometries," *Journal of Mathematical Physics*, vol. 19, no. 7, pp. 1567–1572, 1978.

[19] R. Coquereaux and A. Jadczyk, *Riemannian geometry, fiber bundles, Kaluza-Klein theories and all that...*, vol. 16. World Scientific, 1988.

[20] J. Patera, P. Winternitz, and H. Zassenhaus, "Continuous subgroups of the fundamental groups of physics. i. general method and the Poincaré group," *J. of Mathematical Physics*, vol. 16, pp. 1597–1614, 1975.

[21] H. Flanders, *Differential Forms with Applications to the Physical Sciences*, vol. 11. Elsevier, 1963.

[22] M. E. Fels, "On the construction of simply connected solvable Lie groups," *Journal of Lie Theory*, vol. 27, no. 1, pp. 193–215, 2017.

[23] J. Carminati and R. McLenaghan, "Algebraic invariants of the Riemann tensor in a four-dimensional Lorentzian space," *Journal of mathematical physics*, vol. 32, no. 11, pp. 3135–3140, 1991.

[24] J. Hicks, "The Riemann curvature tensor, its invariants, and their use in the classification of spacetimes," *Presentation at Mathematical Association of America Conference*, March 2015.

APPENDICES

APPENDIX A. RESULTS

A.1 Classification of Lorentzian Lie algebra-subalgebra pairs

[3, 2, 1]

	e_1	e_2	e_3
e_1	.	.	$-e_2$
e_2		.	e_1
e_3			.

Reference : R(3,1), Bowers

Isotropy: $[e_3]$, F12

[3, 2, 2]

	e_1	e_2	e_3
e_1	.	e_1	$-2\,e_2$
e_2		.	e_3
e_3			.

Reference : R(3,2), Bowers

Isotropy: $[\frac{1}{2}\,e_1 - \frac{1}{2}\,e_3]$, F12

[3, 2, 3]

	e_1	e_2	e_3
e_1	.	e_3	$-e_2$
e_2		.	e_1
e_3			.

Reference : R(3,3), Bowers

Isotropy: $[e_3]$, F12

[3, 2, 4]

	e_1	e_2	e_3
e_1	.	.	e_1
e_2		.	$-e_2$
e_3			.

Reference : B(3,1), Bowers

Isotropy: $[e_3]$, F13

[3, 2, 5]

	e_1	e_2	e_3
e_1	.	e_1	$-2\,e_2$
e_2		.	e_3
e_3			.

Reference : B(3,2), Bowers

Isotropy: $[e_2]$, F13

[3, 3, 1]

	e_1	e_2	e_3
e_1	.	$-e_1$	.
e_2		.	.
e_3			.

Reference : s(2,1)+n(1,1), Snobl

Isotropy: F15

[3, 3, 2]

	e_1	e_2	e_3
e_1	.	.	.
e_2		.	.
e_3			.

Reference : 3n(1,1), Snobl

Isotropy: F15

[3, 3, 3]

	e_1	e_2	e_3
e_1	.	.	.
e_2		.	e_1
e_3			.

Reference : n(3,1), Snobl

Isotropy: F15

[3, 3, 4]

	e_1	e_2	e_3
e_1	.	.	$-e_1$
e_2		.	$-a\,e_2$
e_3			.

Reference : s(3,1), Snobl

Parameters: $[a \neq 0, -1 \leq a, a \leq 1]$

Isotropy: F15

[3, 3, 5]

	e_1	e_2	e_3
e_1	.	.	$-e_1$
e_2		.	$-e_2$
e_3			.

Reference : s(3,1) a=1, Snobl

Isotropy: F15

[3, 3, 6]

	e_1	e_2	e_3
e_1	.	.	$-e_1$
e_2		.	$-e_1 - e_2$
e_3			.

REFERENCE : s(3,2), Snobl

ISOTROPY: F15

[3, 3, 7]

	e_1	e_2	e_3
e_1	.	.	$-a\,e_1 + e_2$
e_2		.	$-e_1 - a\,e_2$
e_3			.

REFERENCE : s(3,3), Snobl

PARAMETERS: $[0 < a]$

ISOTROPY: F15

[3, 3, 8]

	e_1	e_2	e_3
e_1	.	$2\,e_1$	$-e_2$
e_2		.	$2\,e_3$
e_3			.

REFERENCE : so(2,1), Snobl

ISOTROPY: F15

[3, 3, 9]

	e_1	e_2	e_3
e_1	.	e_3	$-e_2$
e_2		.	e_1
e_3			.

REFERENCE : so(3), Snobl

ISOTROPY: F15

[4, 3, 1]

	e_1	e_2	e_3	e_4
e_1	.	e_1	$-2\,e_2$	.
e_2		.	e_3	.
e_3			.	.
e_4				.

REFERENCE : R(4,1) b=0, Bowers

ISOTROPY: $[\frac{1}{2}\,e_1 - \frac{1}{2}\,e_3]$, F12

[4, 3, 2]

	e_1	e_2	e_3	e_4
e_1	.	e_1	$-2\,e_2$	.
e_2		.	e_3	.
e_3			.	.
e_4				.

REFERENCE : R(4,1) b=1, Bowers

ISOTROPY: $[\frac{1}{2}\,e_1 - \frac{1}{2}\,e_3 + \frac{1}{2}\,e_4]$, F12

[4, 3, 3]

	e_1	e_2	e_3	e_4
e_1	.	e_3	$-e_2$	.
e_2		.	e_1	.
e_3			.	.
e_4				.

REFERENCE : R(4,2) b=0, Bowers

ISOTROPY: $[\,e_1]$, F12

[4, 3, 4]

	e_1	e_2	e_3	e_4
e_1	.	e_3	$-e_2$	.
e_2		.	e_1	.
e_3			.	.
e_4				.

REFERENCE : R(4,2) b=1, Bowers

ISOTROPY: $[\,e_1 + e_4]$, F12

[4, 3, 5]

	e_1	e_2	e_3	e_4
e_1	.	.	.	.
e_2		.	e_1	$-e_3$
e_3			.	e_2
e_4				.

REFERENCE : R(4,3), Bowers

ISOTROPY: $[\,e_4]$, F12

[4, 3, 6]

	e_1	e_2	e_3	e_4
e_1	.	.	$-e_2$	.
e_2		.	e_1	.
e_3			.	.
e_4				.

REFERENCE : R(4,4), Bowers

ISOTROPY: $[\,e_3]$, F12

[4, 3, 7]

	e_1	e_2	e_3	e_4
e_1	.	.	e_1	$-e_2$
e_2		.	e_2	e_1
e_3			.	.
e_4				.

Reference : R(4,5), Bowers

Isotropy: $[e_4]$, F12

[4, 3, 8]

	e_1	e_2	e_3	e_4
e_1	.	e_1	$-2\,e_2$	.
e_2		.	e_3	.
e_3			.	.
e_4				.

Reference : B(4,1) b=0, Bowers

Isotropy: $[e_2]$, F13

[4, 3, 9]

	e_1	e_2	e_3	e_4
e_1	.	e_1	$-2\,e_2$	.
e_2		.	e_3	.
e_3			.	.
e_4				.

Reference : B(4,1) b=1, Bowers

Isotropy: $[e_2 + e_4]$, F13

[4, 3, 10]

	e_1	e_2	e_3	e_4
e_1	.	.	.	.
e_2		.	e_1	e_2
e_3			.	$-e_3$
e_4				.

Reference : B(4,2), Bowers

Isotropy: $[e_4]$, F13

[4, 3, 11]

	e_1	e_2	e_3	e_4
e_1	.	.	e_1	.
e_2		.	$-e_2$	.
e_3			.	.
e_4				.

Reference : B(4,3), Bowers

Isotropy: $[e_3]$, F13

[4, 3, 12]

	e_1	e_2	e_3	e_4
e_1	.	e_1	.	.
e_2		.	.	.
e_3			.	e_3
e_4				.

Reference : B(4,4), Bowers

Isotropy: $[e_2 - e_4]$, F13

[4, 3, 13]

	e_1	e_2	e_3	e_4
e_1	.	.	.	e_1
e_2		.	e_1	e_2
e_3			.	.
e_4				.

Reference : N(4,1), Bowers

Isotropy: $[e_2 + e_3]$, F14

[4, 3, 14]

	e_1	e_2	e_3	e_4
e_1	.	.	.	$2\,e_1$
e_2		.	e_1	e_2
e_3			.	e_2+e_3
e_4				.

Reference : N(4,2), Bowers

Isotropy: $[e_3]$, F14

[4, 3, 15]

	e_1	e_2	e_3	e_4
e_1	.	.	.	$(a+1)\,e_1$
e_2		.	e_1	e_2
e_3			.	$a\,e_3$
e_4				.

Reference : N(4,3), Bowers

Parameters: $[a \neq 1]$

Isotropy: $[e_2 + e_3]$, F14

[4, 3, 16]

	e_1	e_2	e_3	e_4
e_1	.	.	.	$2\,a\,e_1$
e_2		.	e_1	$a\,e_2-e_3$
e_3			.	$e_2+a\,e_3$
e_4				.

Reference : N(4,4), Bowers

Parameters: $[a \neq 0]$

Isotropy: $[e_2]$, F14

[4, 3, 17]

	e_1	e_2	e_3	e_4
e_1	.	.	.	.
e_2		.	e_1	$-e_3$
e_3			.	e_2
e_4				.

Reference : N(4,5), Bowers

Isotropy: $[e_2]$, F14

[4, 3, 18]

	e_1	e_2	e_3	e_4
e_1	.	.	.	.
e_2		.	e_1	e_2
e_3			.	$-e_3$
e_4				.

Reference : N(4,6), Bowers

Isotropy: $[\frac{1}{2}e_2 + \frac{1}{2}e_3]$, F14

[4, 3, 19]

	e_1	e_2	e_3	e_4
e_1	.	.	.	.
e_2		.	.	e_1
e_3			.	e_2
e_4				.

Reference : N(4,7), Bowers

Isotropy: $[e_4]$, F14

[4, 3, 20]

	e_1	e_2	e_3	e_4
e_1	.	e_1	$-2\,e_2$	.
e_2		.	e_3	.
e_3			.	.
e_4				.

Reference : N(4,8), Bowers

Isotropy: $[e_1 + e_4]$, F14

[4, 4, 1]

	e_1	e_2	e_3	e_4
e_1	.	e_3	$-e_2$	.
e_2		.	e_1	.
e_3			.	.
e_4				.

Reference : so(3)+n(1,1), Snobl

Isotropy: F15

[4, 4, 2]

	e_1	e_2	e_3	e_4
e_1	.	e_1	$2\,e_2$	.
e_2		.	e_3	.
e_3			.	.
e_4				.

Reference : so(2,1)+n(1,1), Snobl

Isotropy: F15

[4, 4, 3]

	e_1	e_2	e_3	e_4
e_1	.	.	$-\alpha\,e_1 + e_2$	.
e_2		.	$-e_1 - \alpha\,e_2$	.
e_3			.	.
e_4				.

Reference : s(3,3)+n(1,1), Snobl

Parameters: $[0 \leq \alpha]$

Isotropy: F15

[4, 4, 4]

	e_1	e_2	e_3	e_4
e_1	.	.	$-e_1$	.
e_2		.	$-e_1 - e_2$	.
e_3			.	.
e_4				.

Reference : s(3,2)+n(1,1), Snobl

Isotropy: F15

[4, 4, 5]

	e_1	e_2	e_3	e_4
e_1	.	.	$-e_1$	.
e_2		.	$-a\,e_2$	.
e_3			.	.
e_4				.

Reference : s(3,1)+n(1,1), Snobl

Parameters: $[0 < \sqrt{a^2}, \sqrt{a^2} \leq 1]$

Isotropy: F15

[4, 4, 6]

	e_1	e_2	e_3	e_4
e_1	.	.	$-e_1$	.
e_2		.	$-e_2$	.
e_3			.	.
e_4				.

Reference : s(3,1)+n(1,1) a=1, Snobl

Isotropy: F15

[4, 4, 7]

	e_1	e_2	e_3	e_4
e_1	.	.	.	.
e_2		.	.	$-e_1$
e_3			.	$-e_3$
e_4				.

Reference : s(4,1), Snobl

Isotropy: F15

[4, 4, 8]

	e_1	e_2	e_3	e_4
e_1	.	$-e_1$	.	.
e_2		.	.	.
e_3			.	$-e_3$
e_4				.

Reference : s(2,1)+s(2,1), Snobl

Isotropy: F15

[4, 4, 9]

	e_1	e_2	e_3	e_4
e_1	.	.	.	.
e_2		.	.	e_1
e_3			.	e_2
e_4				.

Reference : n(4,1), Snobl

Isotropy: F15

[4, 4, 10]

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-e_1$
e_2		.	.	$-a\,e_2$
e_3			.	$-b\,e_3$
e_4				.

Reference : s(4,3), Snobl

Parameters: $[0 < \sqrt{b^2}, \sqrt{b^2} \leq \sqrt{a^2}, a \neq -1, b \neq -1]$

Isotropy: F15

[4, 4, 11]

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-e_1$
e_2		.	.	e_2
e_3			.	$-e_3$
e_4				.

Reference : s(4,3) (a,b)=(-1,1), Snobl

Isotropy: F15

[4, 4, 12]

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-e_1$
e_2		.	.	$-e_1-e_2$
e_3			.	$-a\,e_3$
e_4				.

Reference : s(4,4), Snobl

Parameters: $[a \neq 0]$

Isotropy: F15

[4, 4, 13]

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-e_1$
e_2		.	.	$-e_1-e_2$
e_3			.	$-e_3$
e_4				.

Reference : s(4,4) a=1, Snobl

Isotropy: F15

[4, 4, 14]

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-e_1$
e_2		.	.	$-e_1-e_2$
e_3			.	$-e_2-e_3$
e_4				.

Reference : s(4,2), Snobl

Isotropy: F15

[4, 4, 15]

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-a\,e_1$
e_2		.	.	$-b\,e_2+e_3$
e_3			.	$-e_2-b\,e_3$
e_4				.

Reference : s(4,5), Snobl

Parameters: $[0 < a]$

Isotropy: F15

[4, 4, 16]

	e_1	e_2	e_3	e_4
e_1	.	.	.	$(-a-1)\,e_1$
e_2		.	e_1	$-e_2$
e_3			.	$-a\,e_3$
e_4				.

Reference : s(4,8), Snobl

Parameters: $[-1 < a, a < 1, a \neq 0]$

Isotropy: F15

[4, 4, 17]

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-2\,e_1$
e_2		.	e_1	$-\,e_2$
e_3			.	$-\,e_3$
e_4				.

Reference : s(4,8) a=1, Snobl

Isotropy: F15

[4, 4, 18]

	e_1	e_2	e_3	e_4
e_1	.	.	.	.
e_2		.	e_1	$-\,e_2$
e_3			.	e_3
e_4				.

Reference : s(4,6), Snobl

Isotropy: F15

[4, 4, 19]

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-\,e_1$
e_2		.	e_1	$-\,e_2$
e_3			.	.
e_4				.

Reference : s(4,11), Snobl

Isotropy: F15

[4, 4, 20]

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-2\,e_1$
e_2		.	e_1	$-\,e_2$
e_3			.	$-\,e_2-\,e_3$
e_4				.

Reference : s(4,10), Snobl

Isotropy: F15

[4, 4, 21]

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-2\,a\,e_1$
e_2		.	e_1	$-a\,e_2+e_3$
e_3			.	$-\,e_2-a\,e_3$
e_4				.

Reference : s(4,9), Snobl

Parameters: $[a \neq 0]$

Isotropy: F15

[4, 4, 22]

	e_1	e_2	e_3	e_4
e_1	.	.	.	.
e_2		.	e_1	e_3
e_3			.	$-\,e_2$
e_4				.

Reference : s(4,7), Snobl

Isotropy: F15

[4, 4, 23]

	e_1	e_2	e_3	e_4
e_1	.	.	$-\,e_1$	e_2
e_2		.	$-\,e_2$	$-\,e_1$
e_3			.	.
e_4				.

Reference : s(4,12), Snobl

Isotropy: F15

[5, 4, -6]

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	$2\,e_1$	.
e_2		.	e_1	e_2	e_3
e_3			.	e_3	e_2
e_4				.	.
e_5					.

Reference : A3 epsilon=-1, Fels

Isotropy: $[\,e_3]$, F14, non-reductive

[5, 4, -5]

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	$2\,e_1$	.
e_2		.	e_1	e_2	$-\,e_3$
e_3			.	e_3	e_2
e_4				.	.
e_5					.

Reference : A3 epsilon=1, Fels

Isotropy: $[\,e_3]$, F14, non-reductive

[5, 4, -4]

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$3\,e_1$
e_2		.	.	e_1	$2\,e_2$
e_3			.	e_2	e_3
e_4				.	e_4
e_5					.

Reference : A2 alpha = 2, Fels

Isotropy: $[\,e_4]$, F14, non-reductive

[5, 4, -3]

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$2\,e_1$
e_2		.	.	e_1	e_2
e_3			.	e_2	.
e_4				.	e_4
e_5					.

Reference : A2 alpha =1, Fels

Isotropy: $[\,e_4]$, F14, non-reductive

[5, 4, -2]

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$(\alpha{+}1)\,e_1$
e_2		.	.	e_1	$\alpha\,e_2$
e_3			.	e_2	$(\alpha{-}1)\,e_3$
e_4				.	e_4
e_5					.

Reference : A2, Fels

Parameters: $[\alpha \neq 0, \alpha \neq -1, \alpha \neq 1, \alpha \neq 2]$

Isotropy: $[\,e_4]$, F14, non-reductive

[5, 4, -1]

	e_1	e_2	e_3	e_4	e_5
e_1	.	$2\,e_2$	$-2\,e_3$	.	.
e_2		.	e_1	.	.
e_3			.	.	.
e_4				.	e_4
e_5					.

Reference : A1, Fels

Isotropy: $[\,e_3 + e_4]$, F14, non-reductive

[5, 4, 1]

	e_1	e_2	e_3	e_4	e_5
e_1	.	$2\,e_1$	$-2\,e_2$	.	.
e_2		.	$2\,e_3$	.	.
e_3			.	.	.
e_4				.	.
e_5					.

Reference : (F12, 4), Rozum

Isotropy: $[\,e_1 - e_3 - 2\,e_4]$, F12

[5, 4, 2]

	e_1	e_2	e_3	e_4	e_5
e_1	.	e_3	$-\,e_2$	.	.
e_2		.	e_1	.	.
e_3			.	.	.
e_4				.	.
e_5					.

Reference : (F12, 6), Rozum

Isotropy: $[\,e_1 - e_4]$, F12

[5, 4, 3]

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	e_1	e_3	.
e_3			.	$-\,e_2$	.
e_4				.	.
e_5					.

Reference : (F12, 8), Rozum

Isotropy: $[\,e_4]$, F12

[5, 4, 4]

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	$-2\,e_1$	.
e_2		.	e_1	$-\,e_2$	$-\,e_3$
e_3			.	$-\,e_3$	e_2
e_4				.	.
e_5					.

Reference : (F12, 9), Rozum

Isotropy: $[\,e_5]$, F12

[5, 4, 5]

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	$\beta\,e_1$	.
e_2		.	.	$-\,e_2$	e_3
e_3			.	$-\,e_3$	$-\,e_2$
e_4				.	.
e_5					.

Reference : (F12, 11), Rozum

Parameters: $[\beta \neq 0]$

Isotropy: $[\,e_5]$, F12

[5, 4, 6]

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	$-e_1$	$-e_2$	.
e_3			.	e_3	.
e_4				.	.
e_5					.

Reference : (F13, 3), Rozum

Isotropy: $[e_4]$, F13

[5, 4, 7]

	e_1	e_2	e_3	e_4	e_5
e_1	.	$2e_1$	e_2	.	.
e_2		.	$2e_3$	.	.
e_3			.	.	.
e_4				.	.
e_5					.

Reference : (F13, 5), Rozum

Isotropy: $[e_2 - 2e_4]$, F13

[5, 4, 8]

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	$-e_1$	.
e_2		.	e_1	.	e_2
e_3			.	$-e_3$	$-e_3$
e_4				.	.
e_5					.

Reference : (F13, 6), Rozum

Isotropy: $[e_5]$, F13

[5, 4, 9]

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-e_1$
e_2		.	.	$-e_2$	.
e_3			.	$-a\,e_3$	$-a\,e_3$
e_4				.	.
e_5					.

Reference : (F13, 8), Rozum

Parameters: $[0 < a, a \leq 1]$

Isotropy: $[e_4 - e_5]$, F13

[5, 4, 10]

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	e_1
e_2		.	.	e_1	e_2
e_3			.	e_2	$\epsilon\,e_1 + e_3$
e_4				.	.
e_5					.

Reference : (F14, 1), Rozum

Parameters: $[\epsilon^2 = 1]$

Isotropy: $[e_4]$, F14

[5, 4, 11]

	e_1	e_2	e_3	e_4	e_5
e_1	.	$2e_1$	$-e_2$	.	.
e_2		.	$2e_3$	.	.
e_3			.	.	.
e_4				.	.
e_5					.

Reference : (F14, 2), Rozum

Isotropy: $[e_3 + e_4]$, F14

[6, 3, 1]

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_3	$-e_2$	.	.	.
e_2		.	e_1	.	.	.
e_3			.	.	.	.
e_4				.	e_6	$-e_5$
e_5					.	e_4
e_6						.

Reference : so(3)+so(3), Snobl

Isotropy: $[e_3 - e_6, -e_2 + e_5, e_1 + e_4]$, F3

[6, 3, 2]

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	.	$-e_2$	$-e_3$	.
e_2		.	.	e_1	.	$-e_3$
e_3			.	.	e_1	e_2
e_4				.	e_6	$-e_5$
e_5					.	e_4
e_6						.

Reference : euc(3), Snobl

Isotropy: $[e_4, e_5, e_6]$, F3

[6, 3, 3]

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_3	$-e_2$	e_5	$-e_4$	.
e_2		.	e_1	e_6	.	$-e_4$
e_3			.	.	e_6	$-e_5$
e_4				.	$-e_1$	$-e_2$
e_5					.	$-e_3$
e_6						.

REFERENCE : so(3,1), Snobl

ISOTROPY: $[e_3, -e_1, -e_2]$, F3

[6, 3, 4]

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_2	$-e_3$	.	.	.
e_2		.	$-e_1$	.	.	.
e_3			.	.	.	.
e_4				.	e_5	$-e_6$
e_5					.	$-e_4$
e_6						.

REFERENCE : so(2,1)+so(2,1), Snobl

ISOTROPY:

$[-\frac{1}{2}e_2 - e_3 - \frac{1}{2}e_5 - e_6, \frac{1}{2}e_2 - e_3 - \frac{1}{2}e_5 + e_6, -e_1 + e_4]$
F4

[6, 3, 5]

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	.	$-e_2$	$-e_3$	.
e_2		.	.	e_1	.	$-e_3$
e_3			.	.	$-e_1$	$-e_2$
e_4				.	e_6	$-e_5$
e_5					.	$-e_4$
e_6						.

REFERENCE : euc(2,1), Snobl

ISOTROPY: $[e_4, e_5, e_6]$, F4

[6, 3, 6]

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_3	$-e_2$	e_5	$-e_4$	.
e_2		.	e_1	e_6	.	$-e_4$
e_3			.	.	e_6	$-e_5$
e_4				.	$-e_1$	$-e_2$
e_5					.	$-e_3$
e_6						.

REFERENCE : so(3,1), Snobl

ISOTROPY: $[-e_3, e_6, e_5]$, F4

[6, 4, -1]

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	$2e_2$	$-2e_3$	e_4	$-e_5$	.
e_2		.	e_1	.	e_4	.
e_3			.	e_5	.	.
e_4				.	e_6	.
e_5					.	.
e_6						.

REFERENCE : A4, Fels

ISOTROPY: $[e_3 + e_6, e_5]$, F10, non-reductive

[6, 4, 1]

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_3	$-e_2$	.	.	.
e_2		.	e_1	.	.	.
e_3			.	.	.	.
e_4				.	e_5	$-e_6$
e_5					.	$-e_4$
e_6						.

REFERENCE : so(3)+so(2,1), Snobl

ISOTROPY: $[-e_1, -e_4 - e_5]$, F9

[6, 4, 2]

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_2	$-e_3$	.	.	.
e_2		.	$-e_1$	.	.	.
e_3			.	.	.	.
e_4				.	e_5	$-e_6$
e_5					.	$-e_4$
e_6						.

REFERENCE : so(2,1)+so(2,1), Snobl

ISOTROPY: $[e_4 - e_5 - e_6, -e_1 + e_3]$, F9

[6, 4, 3]

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	e_2	.	.	.
e_2		.	$-e_1$	.	.	.
e_3			.	.	.	.
e_4				.	e_5	$-e_6$
e_5					.	$-e_4$
e_6						.

REFERENCE : s(3,3)(a = 1)+so(2,1), Snobl

ISOTROPY: $[-e_1 - e_2 - e_3, -e_4]$, F9

[6, 4, 4]

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	$-e_1$	.	.	.
e_2		.	e_2	.	.	.
e_3			.	.	.	.
e_4				.	e_6	$-e_5$
e_5					.	e_4
e_6						.

Reference : s(3,1)(a = -1)+so(3), Snobl

Isotropy: $[e_6, \frac{1}{2}e_1 - \frac{1}{2}e_2 - e_3]$, F9

[6, 4, 5]

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	$-e_1$	.	.	.
e_2		.	e_2	.	.	.
e_3			.	.	.	.
e_4				.	e_5	$-e_6$
e_5					.	$-e_4$
e_6						.

Reference : s(3,1)(a = -1)+so(2, 1), Snobl

Isotropy: $[-e_4 + e_5 + e_6, -\frac{1}{2}e_1 - \frac{1}{2}e_2 - e_3]$, F9

[6, 4, 6]

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	.	.	.	$-a\,e_1$
e_2		.	.	e_1	.	$b\,e_3 - e_4$
e_3			.	.	e_1	$-e_5$
e_4				.	.	$c\,e_2 + d\,e_3 - a\,e_4$
e_5					.	$d\,e_2 + f\,e_3 - b\,e_4 - a\,e_5$
e_6						.

Reference : s(158) to s(182), Snobl

Isotropy: $[e_2, e_3]$, F10

[7, 4, 1]

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	e_3	$-e_2$	.	.	.	.
e_2		.	e_1	.	.	.	.
e_3			.	.	.	.	.
e_4				.	e_6	$-e_5$	.
e_5					.	e_4	.
e_6						.	.
e_7							.

Reference : so(3)+so(3)+R, Snobl

Isotropy: $[-e_3 - e_6, e_2 - e_5, e_1 - e_4]$, F3

[7, 4, 2]

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	e_2	e_3	$-e_4$	$-e_5$	.	.
e_2		.	.	$-e_1$	e_6	$-e_3$	.
e_3			.	$-e_6$	$-e_1$	e_2	.
e_4				.	.	$-e_5$	.
e_5					.	e_4	.
e_6						.	.
e_7							.

Reference : so(3,1)+R, Snobl

Isotropy:

$[-e_1 - e_3 - e_5, -e_4 - e_6, e_2 - e_6]$

F3

[7, 4, 3]

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	e_3	$-e_2$	e_5	$-e_4$	.	.
e_2		.	e_1	e_6	.	$-e_4$	.
e_3			.	.	e_6	$-e_5$	.
e_4				.	$-e_1$	$-e_2$	.
e_5					.	$-e_3$	.
e_6						.	.
e_7							.

Reference : so(3,1)+R, Snobl

Isotropy: $[e_3, -e_5, -e_6]$, F4

[7, 4, 4]

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	e_3	$-e_2$	.	.	.	.
e_2		.	$-e_1$	.	.	.	.
e_3			.	.	.	.	.
e_4				.	e_6	$-e_5$	.
e_5					.	$-e_4$	.
e_6						.	.
e_7							.

Reference : so(2,1)+so(2,1)+R

Isotropy: $[-e_1 + e_4, e_3 + e_6, e_2 - e_5]$, F4

[7, 4, 5]

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	.	.	.	.	$-b\,e_1$	.
e_2		.	.	e_1	.	$(1-b)\,e_1-e_4+a\,e_5$	$-2\,e_5$
e_3			.	.	e_1	$(-a+1+\frac{1}{2}\,a\,b)\,e_1-\frac{1}{2}\,a\,b\,e_2+(-b-1)\,e_3+a\,e_4+(-a^2-b-c-1)\,e_5$	$4\,e_1-2\,e_2-2\,e_4+2\,a\,e_5$
e_4				.	.	$-c\,e_1+c\,e_2-b\,e_4+\frac{1}{2}\,a\,b\,e_5$	$(-2-2\,a)\,e_1+2\,a\,e_2+2\,e_3+2\,e_5$
e_5					.	$-e_1+e_3+e_5$	$-2\,e_1+2\,e_2$
e_6						.	.
e_7							.

Reference : 3.3:2, pairs 1,2,3, Komrakov

Isotropy: $[e_7, -e_1+e_2, e_5]$, F6

A.2 Classification of spacetimes with symmetry

A.2.1 G_3 on V_2

A.2.1.1 $F12$

[3, 2, 1]

1. Reference : R(3,1), Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3
Y_1	.	.	$-Y_2$
Y_2		.	Y_1
Y_3			.

	e_1	e_2	e_3
e_1	.	.	$-e_2$
e_2		.	e_1
e_3			.

3. Isomorphisms:

$[X_1 \to -Y_1 + Y_2, X_2 \to Y_1 + Y_2, X_3 \to Y_3]$
$[X_1 \to \frac{1}{2}\,e_1 + \frac{1}{2}\,e_2, X_2 \to -\frac{1}{2}\,e_1 + \frac{1}{2}\,e_2, X_3 \to -e_3]$

4. Isotropy: F12 $[\,e_3]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^1} \quad X_2 = \partial_{x^2} \quad X_3 = -x^2\,\partial_{x^1} + x^1\,\partial_{x^2}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = dx^1\,dx^1 + dx^2\,dx^2 \quad \sigma^3 = \frac{1}{2}\,dx^3\,dx^4$$
$$\sigma^2 = dx^3\,dx^3 \qquad \sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = \frac{1}{4}\,{s_1}^2\left(4\,s_2 s_4 - {s_3}^2\right)$

$\det(g_O) = {s_1}^2$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^3, x^4\right)]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = B\left(x^3, x^4\right), x^4 = x^4]$$
$$\Phi_3 = [x^1 = x^1\xi_1, x^2 = x^2\xi_1, x^3 = x^3, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^3, x^4\right), s_2\left(x^3, x^4\right), 0, e], [e^2 = 1]]$

$[[s_1\left(x^3, x^4\right), 0, 2, 0]]$ missing from Petrov

11. Petrov Reference: [[30, 1, 0]]

[3, 2, 2]

1. Reference : R(3,2), Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3
Y_1	.	$-Y_3$	$-Y_2$
Y_2		.	Y_1
Y_3			.

	e_1	e_2	e_3
e_1	.	e_1	$-2\,e_2$
e_2		.	e_3
e_3			.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3]$
$[X_1 \to -e_2, X_2 \to -\frac{1}{2}\,e_1 - \frac{1}{2}\,e_3, X_3 \to \frac{1}{2}\,e_1 - \frac{1}{2}\,e_3]$

4. Isotropy: F12 $[\frac{1}{2}\,e_1 - \frac{1}{2}\,e_3]$

5. Vector Fields Γ:

$$X_1 = \cosh\left(x^2\right)\,\partial_{x^1} - \frac{\sinh\left(x^2\right)\sinh\left(x^1\right)}{\cosh\left(x^1\right)}\,\partial_{x^2}$$
$$X_2 = \partial_{x^2}$$
$$X_3 = \sinh\left(x^2\right)\,\partial_{x^1} - \frac{\cosh\left(x^2\right)\sinh\left(x^1\right)}{\cosh\left(x^1\right)}\,\partial_{x^2}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1 + \left(\cosh\left(x^1\right)\right)^2\,dx^2\,dx^2 \qquad \sigma^3 = \frac{1}{2}\,dx^3\,dx^4$$
$$\sigma^2 = dx^3\,dx^3 \qquad \sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = \frac{1}{4}\,{s_1}^2\left(\cosh\left(x^1\right)\right)^2\left(4\,s_2s_4 - {s_3}^2\right)$

$\det(g_O) = {s_1}^2\left(\cosh\left(x^1\right)\right)^2$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1\xi_1, x^2 = x^2\xi_1, x^3 = x^3, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^3, x^4\right), s_2\left(x^3, x^4\right), 0, e], [e^2 = 1]]$

$[[s_1\left(x^3, x^4\right), 0, 2, 0]]$ missing from Petrov

11. Petrov Reference: [[30, 3, 0]]

[3, 2, 3]

1. Reference : R(3,3), Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3
Y_1	.	Y_3	$-Y_2$
Y_2		.	Y_1
Y_3			.

	e_1	e_2	e_3
e_1	.	e_3	$-e_2$
e_2		.	e_1
e_3			.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3]$

4. Isotropy: F12 $[e_3]$

5. Vector Fields Γ:

$$X_1 = \cos\left(x^2\right)\,\partial_{x^1} + \frac{\sin\left(x^1\right)\sin\left(x^2\right)}{\cos\left(x^1\right)}\,\partial_{x^2} \quad X_3 = \sin\left(x^2\right)\,\partial_{x^1} - \frac{\cos\left(x^2\right)\sin\left(x^1\right)}{\cos\left(x^1\right)}\,\partial_{x^2}$$
$$X_2 = \partial_{x^2}$$

6. Base Point: $[0,0,0,0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1 + \left(\cos\left(x^1\right)\right)^2\,dx^2\,dx^2 \quad \sigma^3 = \frac{1}{2}\,dx^3\,dx^4$$
$$\sigma^2 = dx^3\,dx^3 \quad \sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = \frac{1}{4}\,{s_1}^2\left(\cos\left(x^1\right)\right)^2\left(4\,s_2s_4 - {s_3}^2\right)$

$\det(g_O) = {s_1}^2\left(\cos\left(x^1\right)\right)^2$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^3, x^4\right)]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = B\left(x^3, x^4\right), x^4 = x^4]$$
$$\Phi_3 = [x^1 = x^1\xi_1, x^2 = x^2\xi_1, x^3 = x^3, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^3, x^4\right), s_2\left(x^3, x^4\right), 0, e], [e^2 = 1]]$

$[[s_1\left(x^3, x^4\right), 0, 2, 0]]$ missing from Petrov

11. Petrov Reference: [[30, 6, 0]]

A.2.1.2 $F13$

[3, 2, 4]

1. Reference : B(3,1), Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3
Y_1	.	.	$-Y_2$
Y_2		.	$-Y_1$
Y_3			.

	e_1	e_2	e_3
e_1	.	.	e_1
e_2		.	$-e_2$
e_3			.

3. Isomorphisms:

$[X_1 \to -Y_1 + Y_2, X_2 \to Y_1 + Y_2, X_3 \to Y_3]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3]$

4. Isotropy: F13 $[e_3]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^1} \quad X_3 = x^1\,\partial_{x^1} - x^2\,\partial_{x^2}$$
$$X_2 = \partial_{x^2}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = dx^1\,dx^2 \quad \sigma^3 = \frac{1}{2}\,dx^3\,dx^4$$
$$\sigma^2 = dx^3\,dx^3 \quad \sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -\frac{1}{4}\,{s_1}^2\left(4\,s_2 s_4 - {s_3}^2\right)$

$\det(g_O) = -{s_1}^2$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^3, x^4\right)]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = B\left(x^3, x^4\right), x^4 = x^4]$$
$$\Phi_3 = [x^1 = x^1 \xi_1, x^2 = x^2, x^3 = x^3, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$$[[s_1\left(x^3, x^4\right), s_2\left(x^3, x^4\right), 0, e], [e^2 = 1]]$$

11. Petrov Reference: [[30, 2, 0]]

[3, 2, 5]

1. Reference : B(3,2), Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3
Y_1	.	$-Y_3$	$-Y_2$
Y_2		.	$-Y_1$
Y_3			.

	e_1	e_2	e_3
e_1	.	e_1	$-2\,e_2$
e_2		.	e_3
e_3			.

3. Isomorphisms:

$[X_1 \to -Y_1, X_2 \to Y_2, X_3 \to Y_3]$
$[X_1 \to \frac{1}{2}\,e_1 + \frac{1}{2}\,e_3, X_2 \to \frac{1}{2}\,e_1 - \frac{1}{2}\,e_3, X_3 \to e_2]$

4. Isotropy: F13 $[\,e_2]$

5. Vector Fields Γ:

$$X_1 = \cos\left(x^2\right)\,\partial_{x^1} - \frac{\sin\left(x^2\right)\sinh\left(x^1\right)}{\cosh\left(x^1\right)}\,\partial_{x^2} \quad X_3 = \sin\left(x^2\right)\,\partial_{x^1} + \frac{\cos\left(x^2\right)\sinh\left(x^1\right)}{\cosh\left(x^1\right)}\,\partial_{x^2}$$
$$X_2 = \partial_{x^2}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1 - \left(\cos\left(x^1\right)\right)^2\,dx^2\,dx^2 \quad \sigma^3 = \frac{1}{2}\,dx^3\,dx^4$$
$$\sigma^2 = dx^3\,dx^3 \quad \sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -\frac{1}{4}\,{s_1}^2\left(\cos\left(x^1\right)\right)^2\left(4\,s_2s_4 - {s_3}^2\right)$

$\det(g_O) = -{s_1}^2\left(\cos\left(x^1\right)\right)^2$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1\xi_1, x^2 = x^2\xi_1, x^3 = x^3, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$$[[s_1\left(x^3, x^4\right), s_2\left(x^3, x^4\right), 0, e], [e^2 = 1]]$$

11. Petrov Reference: [[30, 4, 0], [30, 5, 0]]

A.2.2 G_3 on V_3

[3, 3, 1]

1. Reference : s(2,1)+n(1,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3
Y_1	.	.	Y_1
Y_2		.	.
Y_3			.

	e_1	e_2	e_3
e_1	.	$-e_1$	.
e_2		.	.
e_3			.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3]$
$[X_1 \to e_1, X_2 \to e_3, X_3 \to -e_2]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^2} \quad X_2 = \partial_{x^3} \quad X_3 = -\partial_{x^1} + x^2\,\partial_{x^2}$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1 \qquad \sigma^6 = e^{x^1}\,dx^2\,dx^3$$
$$\sigma^2 = e^{x^1}\,dx^1\,dx^2 \qquad \sigma^7 = \frac{1}{2}\,e^{x^1}\,dx^2\,dx^4$$
$$\sigma^3 = dx^1\,dx^3 \qquad \sigma^8 = dx^3\,dx^3$$
$$\sigma^4 = \frac{1}{2}\,dx^1\,dx^4 \qquad \sigma^9 = \frac{1}{2}\,dx^3\,dx^4$$
$$\sigma^5 = e^{2\,x^1}\,dx^2\,dx^2 \qquad \sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[31, 7, 0], [31, 32, 0], [31, 33, 0], [31, 34, 0], [31, 34, 1], [31, 35, 0]]

[3, 3, 2]

1. Reference : 3n(1,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3
Y_1	.	.	.
Y_2		.	.
Y_3			.

	e_1	e_2	e_3
e_1	.	.	.
e_2		.	.
e_3			.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^1} \quad X_2 = \partial_{x^2} \quad X_3 = \partial_{x^3}$$

5. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = dx^1\,dx^1 \qquad \sigma^6 = dx^2\,dx^3$$
$$\sigma^2 = dx^1\,dx^2 \qquad \sigma^7 = \frac{1}{2}\,dx^2\,dx^4$$
$$\sigma^3 = dx^1\,dx^3 \qquad \sigma^8 = dx^3\,dx^3$$
$$\sigma^4 = \frac{1}{2}\,dx^1\,dx^4 \qquad \sigma^9 = \frac{1}{2}\,dx^3\,dx^4$$
$$\sigma^5 = dx^2\,dx^2 \qquad \sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[31, 5, 0], [31, 27, 0], [31, 28, 0], [31, 29, 0]]

[3, 3, 3]

1. Reference : n(3,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3
Y_1	.	.	.
Y_2		.	Y_1
Y_3			.

	e_1	e_2	e_3
e_1	.	.	.
e_2		.	e_1
e_3			.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^2} \quad X_2 = \partial_{x^3} \quad X_3 = -\partial_{x^1} + x^3\,\partial_{x^2}$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1$$
$$\sigma^2 = dx^1\,dx^2 + x^1\,dx^1\,dx^3$$
$$\sigma^3 = dx^1\,dx^3$$
$$\sigma^4 = \frac{1}{2}\,dx^1\,dx^4$$
$$\sigma^5 = dx^2\,dx^2 + x^1\,dx^2\,dx^3 + {x^1}^2\,dx^3\,dx^3$$
$$\sigma^6 = dx^2\,dx^3 + 2\,x^1\,dx^3\,dx^3$$
$$\sigma^7 = \frac{1}{2}\,dx^2\,dx^4 + x^1/2\,dx^3\,dx^4$$
$$\sigma^8 = dx^3\,dx^3$$
$$\sigma^9 = \frac{1}{2}\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[31, 6, 0], [31, 30, 0], [31, 31, 0], [31, 31, 1]]

[3, 3, 4]

1. Reference : s(3,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3
Y_1	.	.	Y_1
Y_2		.	$a\,Y_2$
Y_3			.

	e_1	e_2	e_3
e_1	.	.	$-e_1$
e_2		.	$-a\,e_2$
e_3			.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to -e_3]$

4. Vector Fields Γ: $[a \neq 0, -1 \leq a, a \leq 1]$

$$X_1 = \partial_{x^2} \quad X_2 = \partial_{x^3} \quad X_3 = -\partial_{x^1} + x^2\,\partial_{x^2} + x^3\,a\,\partial_{x^3}$$

5. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = dx^1\,dx^1 \qquad \sigma^6 = \mathrm{e}^{(1+a)x^1}\,dx^2\,dx^3$$
$$\sigma^2 = \mathrm{e}^{x^1}\,dx^1\,dx^2 \qquad \sigma^7 = \frac{1}{2}\,\mathrm{e}^{x^1}\,dx^2\,dx^4$$
$$\sigma^3 = \mathrm{e}^{a\,x^1}\,dx^1\,dx^3 \qquad \sigma^8 = \mathrm{e}^{2\,a\,x^1}\,dx^3\,dx^3$$
$$\sigma^4 = \frac{1}{2}\,dx^1\,dx^4 \qquad \sigma^9 = \frac{1}{2}\,\mathrm{e}^{a\,x^1}\,dx^3\,dx^4$$
$$\sigma^5 = \mathrm{e}^{2\,x^1}\,dx^2\,dx^2 \qquad \sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[31, 10, 0], [31, 37, 2], [31, 39, 2], [31, 42, 0]]

[3, 3, 5]

1. Reference : s(3,1) a=1, Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3
Y_1	.	.	Y_1
Y_2		.	Y_2
Y_3			.

	e_1	e_2	e_3
e_1	.	.	$-e_1$
e_2		.	$-e_2$
e_3			.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to -e_3]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^2} \quad X_2 = \partial_{x^3} \quad X_3 = -\partial_{x^1} + x^2\,\partial_{x^2} + x^3\,\partial_{x^3}$$

5. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = dx^1\,dx^1 \qquad \sigma^6 = \mathrm{e}^{2\,x^1}\,dx^2\,dx^3$$
$$\sigma^2 = \mathrm{e}^{x^1}\,dx^1\,dx^2 \qquad \sigma^7 = \frac{1}{2}\,\mathrm{e}^{x^1}\,dx^2\,dx^4$$
$$\sigma^3 = \mathrm{e}^{x^1}\,dx^1\,dx^3 \qquad \sigma^8 = \mathrm{e}^{2\,x^1}\,dx^3\,dx^3$$
$$\sigma^4 = \frac{1}{2}\,dx^1\,dx^4 \qquad \sigma^9 = \frac{1}{2}\,\mathrm{e}^{x^1}\,dx^3\,dx^4$$
$$\sigma^5 = \mathrm{e}^{2\,x^1}\,dx^2\,dx^2 \qquad \sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[31, 9, 0], [31, 37, 1], [31, 39, 1], [31, 41, 0]]

[3, 3, 6]

1. Reference : s(3,2), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3
Y_1	.	.	Y_1
Y_2		.	Y_1+Y_2
Y_3			.

	e_1	e_2	e_3
e_1	.	.	$-e_1$
e_2		.	$-e_1-e_2$
e_3			.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to -e_3]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^2} \quad X_2 = \partial_{x^3} \quad X_3 = -\partial_{x^1} + (x^3 + x^2)\,\partial_{x^2} + x^3\,\partial_{x^3}$$

5. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = dx^1\,dx^1$$
$$\sigma^2 = \mathrm{e}^{x^1}\,dx^1\,dx^2 + \mathrm{e}^{x^1}x^1\,dx^1\,dx^3$$
$$\sigma^3 = \mathrm{e}^{x^1}\,dx^1\,dx^3$$
$$\sigma^4 = \frac{1}{2}\,dx^1\,dx^4$$
$$\sigma^5 = \mathrm{e}^{2\,x^1}\,dx^2\,dx^2 + \mathrm{e}^{2\,x^1}x^1\,dx^2\,dx^3 + \mathrm{e}^{2\,x^1}{x^1}^2\,dx^3\,dx^3$$
$$\sigma^6 = \mathrm{e}^{2\,x^1}\,dx^2\,dx^3 + 2\,\mathrm{e}^{2\,x^1}x^1\,dx^3\,dx^3$$
$$\sigma^7 = \frac{1}{2}\,\mathrm{e}^{x^1}\,dx^2\,dx^4 + \frac{1}{2}\,\mathrm{e}^{x^1}x^1\,dx^3\,dx^4$$
$$\sigma^8 = \mathrm{e}^{2\,x^1}\,dx^3\,dx^3$$
$$\sigma^9 = \frac{1}{2}\,\mathrm{e}^{x^1}\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[31, 8, 0], [31, 37, 0], [31, 39, 0], [31, 40, 0]]

[3, 3, 7]

1. Reference : s(3,3), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3
Y_1	.	.	$-a\,Y_1+Y_2$
Y_2		.	$-a\,Y_2-Y_1$
Y_3			.

	e_1	e_2	e_3
e_1	.	.	$-a\,e_1+e_2$
e_2		.	$-a\,e_2-e_1$
e_3			.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3]$

4. Vector Fields Γ: $[0 < a]$

$$X_1 = \partial_{x^1} \quad X_2 = \partial_{x^2} \quad X_3 = (-a\,x^1 - x^2)\,\partial_{x^1} + (-a\,x^2 + x^1)\,\partial_{x^2} + \partial_{x^3}$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = -e^{-2\,x^3\,a}\sin\left(2\,x^3\right)\,dx^1\,dx^1 - e^{-2\,x^3\,a}\cos\left(2\,x^3\right)\,dx^1\,dx^2 + e^{-2\,x^3\,a}\sin\left(2\,x^3\right)\,dx^2\,dx^2$$
$$\sigma^2 = e^{-2\,x^3\,a}\,dx^1\,dx^1 + e^{-2\,x^3\,a}\,dx^2\,dx^2$$
$$\sigma^3 = -e^{-2\,x^3\,a}\cos\left(2\,x^3\right)\,dx^1\,dx^1 + e^{-2\,x^3\,a}\sin\left(2\,x^3\right)\,dx^1\,dx^2 + e^{-2\,x^3\,a}\cos\left(2\,x^3\right)\,dx^2\,dx^2$$
$$\sigma^4 = \frac{1}{2}\,e^{-x^3\,a}\sin\left(x^3\right)\,dx^1\,dx^3 + \frac{1}{2}\,e^{-x^3\,a}\cos\left(x^3\right)\,dx^2\,dx^3$$
$$\sigma^5 = -\frac{1}{2}\,e^{-x^3\,a}\cos\left(x^3\right)\,dx^1\,dx^3 + \frac{1}{2}\,e^{-x^3\,a}\sin\left(x^3\right)\,dx^2\,dx^3$$
$$\sigma^6 = \frac{1}{2}\,e^{-x^3\,a}\sin\left(x^3\right)\,dx^1\,dx^4 + \frac{1}{2}\,e^{-x^3\,a}\cos\left(x^3\right)\,dx^2\,dx^4$$
$$\sigma^7 = -\frac{1}{2}\,e^{-x^3\,a}\cos\left(x^3\right)\,dx^1\,dx^4 + \frac{1}{2}\,e^{-x^3\,a}\sin\left(x^3\right)\,dx^2\,dx^4$$
$$\sigma^8 = dx^3\,dx^3$$
$$\sigma^9 = \frac{1}{2}\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[31, 11, 0], [31, 43, 0], [31, 44, 0]]

[3, 3, 8]

1. Reference : so(2,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3
Y_1	.	Y_1	$2\,Y_2$
Y_2		.	Y_3
Y_3			.

	e_1	e_2	e_3
e_1	.	$2\,e_1$	$-e_2$
e_2		.	$2\,e_3$
e_3			.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3]$
$[X_1 \to \frac{1}{2}\,e_1, X_2 \to \frac{1}{2}\,e_2, X_3 \to -2\,e_3]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^2} \quad X_2 = x^2\,\partial_{x^2} + \partial_{x^3} \quad X_3 = \mathrm{e}^{x^3}\,\partial_{x^1} + x^4\,\partial_{x^2} + 2\,x^2\,\partial_{x^3}$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1 + {x^1}^2\mathrm{e}^{-x^3}\,dx^1\,dx^2 - x^1\,dx^1\,dx^3 + {x^1}^4\mathrm{e}^{-2\,x^3}\,dx^2\,dx^2$$
$$- {x^1}^3\mathrm{e}^{-x^3}\,dx^2\,dx^3 + {x^1}^2\,dx^3\,dx^3$$
$$\sigma^2 = \mathrm{e}^{-x^3}\,dx^1\,dx^2 + 2\,\mathrm{e}^{-2\,x^3}{x^1}^2\,dx^2\,dx^2 - x^1\,\mathrm{e}^{-x^3}\,dx^2\,dx^3$$
$$\sigma^3 = -2\,\mathrm{e}^{-x^3}\,dx^1\,dx^2 + dx^3\,dx^3$$
$$\sigma^4 = 2\,x^1\,\mathrm{e}^{-x^3}\,dx^1\,dx^2 - dx^1\,dx^3 + 4\,{x^1}^3\mathrm{e}^{-2\,x^3}\,dx^2\,dx^2 - 3\,{x^1}^2\mathrm{e}^{-x^3}\,dx^2\,dx^3$$
$$+ 2\,x^1\,dx^3\,dx^3$$
$$\sigma^5 = -\frac{1}{2}\,dx^1\,dx^4 - \frac{1}{2}\,{x^1}^2\mathrm{e}^{-x^3}\,dx^2\,dx^4 + x^1/2\,dx^3\,dx^4$$
$$\sigma^6 = -4\,\mathrm{e}^{-2\,x^3}x^1\,dx^2\,dx^2 + \mathrm{e}^{-x^3}\,dx^2\,dx^3$$
$$\sigma^7 = \mathrm{e}^{-2\,x^3}\,dx^2\,dx^2$$
$$\sigma^8 = \frac{1}{2}\,\mathrm{e}^{-x^3}\,dx^2\,dx^4$$
$$\sigma^9 = -x^1\,\mathrm{e}^{-x^3}\,dx^2\,dx^4 + \frac{1}{2}\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[31, 14, 0], [31, 45, 0], [31, 46, 0], [31, 47, 0]]

[3, 3, 9]

1. Reference : so(3), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3
Y_1	.	Y_3	$-Y_2$
Y_2		.	Y_1
Y_3			.

	e_1	e_2	e_3
e_1	.	e_3	$-e_2$
e_2		.	e_1
e_3			.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^2}$$
$$X_2 = \cos\left(x^2\right)\,\partial_{x^1} - \frac{\cos\left(x^1\right)\sin\left(x^2\right)}{\sin\left(x^1\right)}\,\partial_{x^2} + \frac{\sin\left(x^2\right)}{\sin\left(x^1\right)}\,\partial_{x^3}$$
$$X_3 = -\sin\left(x^2\right)\,\partial_{x^1} - \frac{\cos\left(x^1\right)\cos\left(x^2\right)}{\sin\left(x^1\right)}\,\partial_{x^2} + \frac{\cos\left(x^2\right)}{\sin\left(x^1\right)}\,\partial_{x^3}$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = -\sin\left(2\,x^3\right)\,dx^1\,dx^1 + \cos\left(2\,x^3\right)\sin\left(x^1\right)\,dx^1\,dx^2 + \sin\left(2\,x^3\right)\left(\sin\left(x^1\right)\right)^2\,dx^2\,dx^2$$
$$\sigma^2 = dx^1\,dx^1 + \left(\sin\left(x^1\right)\right)^2\,dx^2\,dx^2$$
$$\sigma^3 = dx^1\,dx^1 + dx^2\,dx^2 + \cos\left(x^1\right)\,dx^2\,dx^3 + dx^3\,dx^3$$
$$\sigma^4 = -\cos\left(2\,x^3\right)\,dx^1\,dx^1 - \sin\left(2\,x^3\right)\sin\left(x^1\right)\,dx^1\,dx^2 + \cos\left(2\,x^3\right)\left(\sin\left(x^1\right)\right)^2\,dx^2\,dx^2$$
$$\sigma^5 = -\frac{1}{2}\cos\left(x^1\right)\sin\left(x^3\right)\,dx^1\,dx^2 - \frac{1}{2}\sin\left(x^3\right)\,dx^1\,dx^3 + \cos\left(x^1\right)\cos\left(x^3\right)\sin\left(x^1\right)\,dx^2\,dx^2 + \frac{1}{2}\cos\left(x^3\right)\sin\left(x^1\right)\,dx^2\,dx^3$$
$$\sigma^6 = \frac{1}{2}\cos\left(x^1\right)\cos\left(x^3\right)\,dx^1\,dx^2 + \frac{1}{2}\cos\left(x^3\right)\,dx^1\,dx^3 + \cos\left(x^1\right)\sin\left(x^3\right)\sin\left(x^1\right)\,dx^2\,dx^2 + \frac{1}{2}\sin\left(x^3\right)\sin\left(x^1\right)\,dx^2\,dx^3$$
$$\sigma^7 = -\frac{1}{2}\sin\left(x^3\right)\,dx^1\,dx^4 + \frac{1}{2}\cos\left(x^3\right)\sin\left(x^1\right)\,dx^2\,dx^4$$
$$\sigma^8 = \frac{1}{2}\cos\left(x^3\right)\,dx^1\,dx^4 + \frac{1}{2}\sin\left(x^3\right)\sin\left(x^1\right)\,dx^2\,dx^4$$
$$\sigma^9 = \frac{1}{2}\cos\left(x^1\right)\,dx^2\,dx^4 + \frac{1}{2}\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[31, 15, 0], [31, 48, 0]]

A.2.3 G_4 on V_3

A.2.3.1 $F12$

[4, 3, 1]

1. Reference : R(4,1) b=0, Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	$-12\,Y_4$	.	$-Y_2$
Y_2		.	.	Y_1
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	e_1	$-2\,e_2$	.
e_2		.	e_3	.
e_3			.	.
e_4				.

3. Isomorphisms:

$$X_1 \to -1/6\,\sqrt{3}\,Y_1 - 1/6\,Y_2 + 2/3\,\sqrt{3}\,Y_4, \quad X_3 \to -1/6\,\sqrt{3}\,Y_1 + 1/6\,Y_2 - 2/3\,\sqrt{3}\,Y_4,$$
$$X_2 \to 1/3\,Y_2 - 1/3\,\sqrt{3}\,Y_4, \quad X_4 \to 1/3\,Y_3$$

$$X_1 \to 2/3\,\sqrt{3}\,e_3, \quad X_3 \to \frac{1}{2}\,\sqrt{3}\,e_1 - \,e_2 - 1/6\,\sqrt{3}\,e_3,$$
$$X_2 \to -\,e_2 - 1/3\,\sqrt{3}\,e_3, \quad X_4 \to \,e_4$$

4. Isotropy: F12 $[\frac{1}{2}\,e_1 - \frac{1}{2}\,e_3]$

5. Vector Fields Γ:

$$X_1 = \cos\left(x^3\right)\,\partial_{x^2} - \frac{\cosh\left(x^2\right)\sin\left(x^3\right) - \sinh\left(x^2\right)}{\sinh\left(x^2\right)}\,\partial_{x^3}$$

$$X_2 = \sin\left(x^3\right)\,\partial_{x^2} + \frac{\cosh\left(x^2\right)\cos\left(x^3\right)}{\sinh\left(x^2\right)}\,\partial_{x^3}$$

$$X_3 = \cos\left(x^3\right)\,\partial_{x^2} - \frac{\cosh\left(x^2\right)\sin\left(x^3\right) + \sinh\left(x^2\right)}{\sinh\left(x^2\right)}\,\partial_{x^3}$$

$$X_4 = \,\partial_{x^1}$$

6. Base Point: $[0, -\mathrm{arctanh}\left(\frac{1}{2}\right), 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = \,dx^1\,dx^1 \quad \sigma^3 = \,dx^2\,dx^2 + (-\frac{1}{2} + \frac{1}{2}\,\cosh\left(2\,x^2\right))\,dx^3\,dx^3$$
$$\sigma^2 = \frac{1}{2}\,dx^1\,dx^4 \quad \sigma^4 = \,dx^4\,dx^4$$

8. Determinants :

$\det(g) = \frac{1}{4}\,{s_3}^2\left(\sinh\left(x^2\right)\right)^2\left(4\,s_1 s_4 - {s_2}^2\right)$

$\det(g_O) = s_1 {s_3}^2\left(\sinh\left(x^2\right)\right)^2$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1 + B\left(x^4\right), x^2 = x^2, x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^4\right)]$$
$$\Phi_3 = [x^1 = x^1 \xi_1, x^2 = x^2, x^3 = x^3, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^4\right), 0, s_3\left(x^4\right), e], [e^2 = 1]]$	timelike and spacelike orbits
$[[0, 2, s_3\left(x^4\right), 0]]$	null orbits

11. Petrov Reference: [[32, 7, 0], [32, 23, 0]]

[4, 3, 2]

1. Reference : R(4,1) b=1, Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	$-Y_3$	$-Y_2$	$-Y_2$
Y_2		.	Y_1	Y_1
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	e_1	$-2\,e_2$	.
e_2		.	e_3	.
e_3			.	.
e_4				.

3. Isomorphisms:

$[X_1 \to -Y_1 + Y_3, X_2 \to Y_2, X_3 \to Y_1 + Y_3, X_4 \to -2\,Y_3 + 2\,Y_4]$
$[X_1 \to -e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to -e_4]$

4. Isotropy: F12 $[\frac{1}{2}\,e_1 - \frac{1}{2}\,e_3 + \frac{1}{2}\,e_4]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2} \qquad X_3 = -\mathrm{e}^{x^3}\,\partial_{x^1} + (-\mathrm{e}^{2\,x^3} + x^4)\,\partial_{x^2} + 2\,x^2\,\partial_{x^3}$$
$$X_2 = x^2\,\partial_{x^2} + \partial_{x^3} \qquad X_4 = \partial_{x^1}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = 4\,dx^1\,dx^1 - 2\,\mathrm{e}^{-x^3}\,dx^1\,dx^2 + \mathrm{e}^{-2\,x^3}\,dx^2\,dx^2$$
$$\sigma^2 = -4\,dx^1\,dx^1 + 2\,\mathrm{e}^{-x^3}\,dx^1\,dx^2 + dx^3\,dx^3$$
$$\sigma^3 = -\,dx^1\,dx^4 + \frac{1}{2}\,\mathrm{e}^{-x^3}\,dx^2\,dx^4$$
$$\sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = \mathrm{e}^{-2\,x^3} {s_2}^2 \left(4\,s_1 s_4 - 4\,s_2 s_4 - {s_3}^2\right)$

$\det(g_O) = 4\,\mathrm{e}^{-2\,x^3} {s_2}^2\,(s_1 - s_2)$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1 + B\left(x^4\right), x^2 = x^2, x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^4\right)]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^4\right), s_2\left(x^4\right), 0, e], [e^2 = 1]]$	timelike and spacelike orbits, missing from Petrov
$[[s_2\left(x^4\right), s_2\left(x^4\right), -1, 0]]$	null orbits

11. Petrov Reference: [[32, 24, 0]]

[4, 3, 3]

1. Reference : R(4,2) b=0, Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	Y_4	.	$-Y_2$
Y_2		.	.	Y_1
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	e_3	$-e_2$	.
e_2		.	e_1	.
e_3			.	.
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_4, X_4 \to Y_3]$
$[X_1 \to e_3, X_2 \to -e_2, X_3 \to e_1, X_4 \to e_4]$

4. Isotropy: F12 $[e_1]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^3} \qquad X_3 = \cos\left(x^3\right)\partial_{x^2} - \frac{\sin\left(x^3\right)\cos\left(x^2\right)}{\sin\left(x^2\right)}\partial_{x^3}$$

$$X_2 = \sin\left(x^3\right)\partial_{x^2} + \frac{\cos\left(x^2\right)\cos\left(x^3\right)}{\sin\left(x^2\right)}\partial_{x^3} \qquad X_4 = \partial_{x^1}$$

6. Base Point: $[0, \pi/2, \pi/2, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1 \qquad \sigma^3 = dx^2\,dx^2 + (\frac{1}{2} - \frac{1}{2}\cos\left(2\,x^2\right))\,dx^3\,dx^3$$

$$\sigma^2 = \frac{1}{2}\,dx^1\,dx^4 \qquad \sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = \frac{1}{4}\,{s_3}^2\left(\sin\left(x^2\right)\right)^2\left(4\,s_1 s_4 - {s_2}^2\right)$

$\det(g_O) = s_1 {s_3}^2\left(\sin\left(x^2\right)\right)^2$

9. Normalizers:

$\Phi_1 = [x^1 = x^1 + B\left(x^4\right), x^2 = x^2, x^3 = x^3, x^4 = x^4]$
$\Phi_2 = [x^1 = x^1\xi_1, x^2 = x^2, x^3 = x^3, x^4 = x^4]$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^4\right), 0, s_3\left(x^4\right), e], [e^2 = 1]]$ timelike and spacelike orbits
$[[0, 1, s_3\left(x^4\right), s_4\left(x^4\right)]]$ null orbits

11. Petrov Reference: [[32, 9, 0], [32, 25, 0]]

[4, 3, 4]

1. Reference : R(4,2) b=1, Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	$-Y_3$	Y_2	Y_2
Y_2		.	$-Y_1$	$-Y_1$
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	e_3	$-e_2$	.
e_2		.	e_1	.
e_3			.	.
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_3, X_3 \to Y_2, X_4 \to -Y_3 + Y_4]$
$[X_1 \to e_2, X_2 \to -e_1, X_3 \to e_3, X_4 \to -e_4]$

4. Isotropy: F12 $[e_1 + e_4]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2}$$
$$X_2 = \frac{\cos(x^2)}{\cos(x^3)}\,\partial_{x^1} - \frac{\sin(x^3)\cos(x^2)}{\cos(x^3)}\,\partial_{x^2} + \sin(x^2)\,\partial_{x^3}$$
$$X_3 = -\frac{\sin(x^2)}{\cos(x^3)}\,\partial_{x^1} + \frac{\sin(x^3)\sin(x^2)}{\cos(x^3)}\,\partial_{x^2} + \cos(x^2)\,\partial_{x^3}$$
$$X_4 = \partial_{x^1}$$

6. Base Point: $[0, 0, \pi, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1 + \sin(x^3)\,dx^1\,dx^2 + (\frac{1}{2} - \frac{1}{2}\cos(2\,x^3))\,dx^2\,dx^2$$
$$\sigma^2 = dx^1\,dx^1 + \sin(x^3)\,dx^1\,dx^2 + dx^2\,dx^2 + dx^3\,dx^3$$
$$\sigma^3 = \frac{1}{2}\,dx^1\,dx^4 + \frac{1}{2}\sin(x^3)\,dx^2\,dx^4$$
$$\sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = \frac{1}{4}\,{s_2}^2\left(\cos(x^3)\right)^2\left(4\,s_4 s_1 + 4\,s_2 s_4 - {s_3}^2\right)$

$\det(g_O) = {s_2}^2\left(\cos(x^3)\right)^2(s_1 + s_2)$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1 + B(x^4), x^2 = x^2, x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A(x^4)]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1(x^4), s_2(x^4), 0, e], [e^2 = 1]]$	timelike and spacelike orbits
$[[-s_2(x^4), s_2(x^4), 2, 0]]$	null orbits

11. Petrov Reference: [[32, 10, 0], [32, 25, 1]]

[4, 3, 5]

1. Reference : R(4,3), Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	$-Y_3$	.	$-Y_2$
Y_2		.	.	Y_1
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	.
e_2		.	e_1	$-e_3$
e_3			.	e_2
e_4				.

3. Isomorphisms:

$[X_1 \to Y_3, X_2 \to Y_2, X_3 \to Y_1, X_4 \to Y_4]$
$[X_1 \to \frac{1}{2}\, e_1, X_2 \to -\frac{1}{2}\, e_2 - \frac{1}{2}\, e_3, X_3 \to \frac{1}{2}\, e_2 - \frac{1}{2}\, e_3, X_4 \to -\, e_4]$

4. Isotropy: F12 $[\, e_4]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2} \quad X_3 = -\,\partial_{x^1} + x^3\,\partial_{x^2}$$
$$X_2 = \partial_{x^3} \quad X_4 = -x^3\,\partial_{x^1} + (\frac{1}{2}\,x^6 - \frac{1}{2}\,\left(x^1\right)^2)\,\partial_{x^2} + x^1\,\partial_{x^3}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = dx^1\,dx^1 + dx^3\,dx^3 \qquad \sigma^3 = \frac{1}{2}\,dx^2\,dx^4 + \frac{x^1}{2}\,dx^3\,dx^4$$
$$\sigma^2 = dx^2\,dx^2 + x^1\,dx^2\,dx^3 + {x^1}^2\,dx^3\,dx^3 \qquad \sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = \frac{1}{4}\, {s_1}^2 \left(4\, s_2 s_4 - {s_3}^2\right)$

$\det(g_O) = s_2 {s_1}^2$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1, x^2 = x^2 + B\left(x^4\right), x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^4\right)]$$
$$\Phi_3 = [x^1 = x^1 \xi_1, x^2 = x^2 {\xi_1}^2, x^3 = x^3 \xi_1, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^4\right), s_2\left(x^4\right), 0, e], [e^2 = 1]]$ timelike and spacelike orbits

$[[s_1\left(x^4\right), 0, 2\,e, 0], [e^2 = 1]]$ null orbits, isometry dimension 6, non-simple G

11. Petrov Reference: [[32, 4, 0], [32, 20, 0]]

[4, 3, 6]

1. Reference : R(4,4), Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	$-Y_2$
Y_2		.	.	Y_1
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	$-e_2$	.
e_2		.	e_1	.
e_3			.	.
e_4				.

3. Isomorphisms:

$[X_1 \to -Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to -\frac{1}{2}\, e_1 + \frac{1}{2}\, e_2, X_2 \to \frac{1}{2}\, e_1 + \frac{1}{2}\, e_2, X_3 \to e_4, X_4 \to e_3]$

4. Isotropy: F12 $[\, e_3]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2} \quad X_3 = -\,\partial_{x^1}$$
$$X_2 = \partial_{x^3} \quad X_4 = -x^3\,\partial_{x^2} + x^2\,\partial_{x^3}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = dx^1\,dx^1 \qquad \sigma^3 = dx^2\,dx^2 + dx^3\,dx^3$$
$$\sigma^2 = \frac{1}{2}\,dx^1\,dx^4 \quad \sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = \frac{1}{4}\, {s_3}^2 \left(4\, s_4 s_1 - {s_2}^2\right)$

$\det(g_O) = {s_3}^2 s_1$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1 + B\left(x^4\right), x^2 = x^2, x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^4\right)]$$
$$\Phi_3 = [x^1 = x^1\xi_1, x^2 = x^2\xi_2, x^3 = \xi_2 x^3, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^4\right), 0, s_3\left(x^4\right), e], [e^2 = 1]]$	timelike and spacelike orbits
$[[0, 2\,e, s_3\left(x^4\right), 0], [e^2 = 1]]$	null orbits, isometry dimension 6, non-simple G

11. Petrov Reference: [[32, 11, 1], [32, 27, 1]]

[4, 3, 7]

1. Reference : R(4,5), Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	Y_1	$-Y_2$
Y_2		.	Y_2	Y_1
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	e_1	$-e_2$
e_2		.	e_2	e_1
e_3			.	.
e_4				.

3. Isomorphisms:

$[X_1 \to -Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to \frac{1}{2}\, e_1 + \frac{1}{2}\, e_2, X_2 \to -\frac{1}{2}\, e_1 + \frac{1}{2}\, e_2, X_3 \to e_3, X_4 \to -e_4]$

4. Isotropy: F12 $[\, e_4]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2} \quad X_3 = -\partial_{x^1} + x^2\,\partial_{x^2} + x^3\,\partial_{x^3}$$
$$X_2 = \partial_{x^3} \quad X_4 = -x^3\,\partial_{x^2} + x^2\,\partial_{x^3}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = dx^1\,dx^1 \qquad \sigma^3 = \mathrm{e}^{2\,x^1}\,dx^2\,dx^2 + \mathrm{e}^{2\,x^1}\,dx^3\,dx^3$$
$$\sigma^2 = \frac{1}{2}\,dx^1\,dx^4 \quad \sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = \frac{1}{4}\, {s_3}^2 \mathrm{e}^{4\,x^1} \left(4\, s_4 s_1 - {s_2}^2\right)$

$\det(g_O) = s_1 {s_3}^2 \mathrm{e}^{4\,x^1}$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1 + B\left(x^4\right), x^2 = x^2, x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^4\right)]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^4\right), 0, s_3\left(x^4\right), e], [e^2 = 1]]$ timelike and spacelike orbits
$[[0, 2, s_3\left(x^4\right), 0]]$ null orbits

11. Petrov Reference: [[32, 6, 0], [32, 22, 0]]

A.2.3.2 $F13$

[4, 3, 8]

1. Reference : B(4,1) b=0, Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	.
Y_2		.	$-Y_4$	$-Y_3$
Y_3			.	$-Y_2$
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	e_1	$-2\,e_2$	.
e_2		.	e_3	.
e_3			.	.
e_4				.

3. Isomorphisms:

$[X_1 \to Y_3 + Y_4, X_2 \to Y_2, X_3 \to Y_3 - Y_4, X_4 \to Y_1]$
$[X_1 \to e_1 + e_2 - \frac{1}{4}\,e_3, X_2 \to -e_1 - \frac{1}{4}\,e_3, X_3 \to e_1 - e_2 - \frac{1}{4}\,e_3, X_4 \to e_4]$

4. Isotropy: F13 $[\,e_2]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2} \qquad X_3 = (\mathrm{e}^{2\,x^3} + x^4)\,\partial_{x^2} + 2\,x^2\,\partial_{x^3}$$
$$X_2 = x^2\,\partial_{x^2} + \partial_{x^3} \qquad X_4 = \partial_{x^1}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1 \qquad \sigma^3 = \mathrm{e}^{-2\,x^3}\,dx^2\,dx^2 - dx^3\,dx^3$$
$$\sigma^2 = \frac{1}{2}\,dx^1\,dx^4 \qquad \sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -\frac{1}{4}\,{s_3}^2\mathrm{e}^{-2\,x^3}\left(4\,s_4 s_1 - {s_2}^2\right)$

$\det(g_O) = -s_1 {s_3}^2\mathrm{e}^{-2\,x^3}$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1 + B\left(x^4\right), x^2 = x^2, x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^4\right)]$$
$$\Phi_3 = [x^1 = x^1\xi_1, x^2 = x^2, x^3 = x^3, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^4\right), 0, s_3\left(x^4\right), e], [e^2 = 1]]$	timelike and spacelike orbits, missing from Petrov
$[[0, 2\,e, s_3\left(x^4\right), 0], [e^2 = 1]]$	null orbits and non-Lorentzian signature

11. Petrov Reference: [[32, 23, 1]]

[4, 3, 9]

1. Reference : B(4,1) b=1, Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	$-Y_3$	$-Y_2$	.
Y_2		.	Y_1	$-Y_3$
Y_3			.	$-Y_2$
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	e_1	$-2\,e_2$	.
e_2		.	e_3	.
e_3			.	.
e_4				.

3. Isomorphisms:

$[X_1 \to -Y_1 + Y_3, X_2 \to Y_2, X_3 \to Y_1 + Y_3, X_4 \to -2\,Y_1 - 2\,Y_4]$
$[X_1 \to -e_1 + e_2 + \frac{1}{4}\,e_3, X_2 \to e_1 + \frac{1}{4}\,e_3, X_3 \to -e_1 - e_2 + \frac{1}{4}\,e_3, X_4 \to -2\,e_4]$

4. Isotropy: F13 $[\,e_2 + e_4]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2}$$
$$X_2 = x^2\,\partial_{x^2} + \partial_{x^3}$$
$$X_3 = -\mathrm{e}^{x^3}\,\partial_{x^1} + (\mathrm{e}^{2\,x^3} + x^4)\,\partial_{x^2} + 2\,x^2\,\partial_{x^3}$$
$$X_4 = \partial_{x^1}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = 4\,dx^1\,dx^1 + 2\,\mathrm{e}^{-x^3}\,dx^1\,dx^2 + \mathrm{e}^{-2\,x^3}\,dx^2\,dx^2$$
$$\sigma^2 = 4\,dx^1\,dx^1 + 2\,\mathrm{e}^{-x^3}\,dx^1\,dx^2 + dx^3\,dx^3$$
$$\sigma^3 = dx^1\,dx^4 + \frac{1}{2}\,\mathrm{e}^{-x^3}\,dx^2\,dx^4$$
$$\sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -\mathrm{e}^{-2\,x^3}{s_2}^2\left(4\,s_4s_1 + 4\,s_2s_4 - {s_3}^2\right)$

$\det(g_O) = -4\,\mathrm{e}^{-2\,x^3}{s_2}^2\,(s_1 + s_2)$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1 + B\left(x^4\right), x^2 = x^2, x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^4\right)]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^4\right), s_2\left(x^4\right), 0, e], [e^2 = 1]]$	timelike and spacelike orbits, missing from Petrov
$[[s_1\left(x^4\right), -s_1\left(x^4\right), 1, 0]]$	null orbits and non-Lorentzian signature

11. Petrov Reference: [[32, 24, 1]]

[4, 3, 10]

1. Reference : B(4,2), Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	.
Y_2		.	$-\frac{1}{2}\,Y_1$	$-Y_3$
Y_3			.	$-Y_2$
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	.
e_2		.	e_1	e_2
e_3			.	$-e_3$
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to -Y_2 + Y_3, X_3 \to Y_2 + Y_3, X_4 \to Y_4]$
$[X_1 \to -e_1, X_2 \to -e_3, X_3 \to -e_2, X_4 \to -e_4]$

4. Isotropy: F13 $[\,e_4]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2} \quad X_3 = -\partial_{x^1} + x^3\,\partial_{x^2}$$
$$X_2 = \partial_{x^3} \quad X_4 = -x^1\,\partial_{x^1} + x^3\,\partial_{x^3}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = dx^1\,dx^3 \qquad \sigma^3 = \frac{1}{2}\,dx^2\,dx^4 + x^1/2\,dx^3\,dx^4$$
$$\sigma^2 = dx^2\,dx^2 + x^1\,dx^2\,dx^3 + {x^1}^2\,dx^3\,dx^3 \qquad \sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -\frac{1}{4}\,{s_1}^2\left(4\,s_2 s_4 - {s_3}^2\right)$

$\det(g_O) = -s_2 {s_1}^2$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1, x^2 = x^2 + B\left(x^4\right), x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^4\right)]$$
$$\Phi_3 = [x^1 = x^1\xi_1, x^2 = x^2\xi_1, x^3 = x^3, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^4\right), s_2\left(x^4\right), 0, e], [e^2 = 1]]$	timelike and spacelike orbits
$[[s_1\left(x^4\right), 0, 2, 0]]$	null orbits, non-Lorentzian signature, isometry dim 6

11. Petrov Reference: [[32, 3, 0], [32, 18, 0]]

[4, 3, 11]

1. Reference : B(4,3), Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	.
Y_2		.	.	$-Y_3$
Y_3			.	$-Y_2$
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	e_1	.
e_2		.	$-e_2$	.
e_3			.	.
e_4				.

3. Isomorphisms:

$[X_1 \to Y_3, X_2 \to -Y_2, X_3 \to Y_1, X_4 \to Y_4]$
$[X_1 \to \frac{1}{2}\,e_1 + \frac{1}{2}\,e_2, X_2 \to \frac{1}{2}\,e_1 - \frac{1}{2}\,e_2, X_3 \to -e_4, X_4 \to e_3]$

4. Isotropy: F13 $[\,e_3]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2} \quad X_3 = -\partial_{x^1}$$
$$X_2 = \partial_{x^3} \quad X_4 = x^3\,\partial_{x^2} + x^2\,\partial_{x^3}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1 \qquad \sigma^3 = dx^2\,dx^2 - dx^3\,dx^3$$
$$\sigma^2 = \frac{1}{2}\,dx^1\,dx^4 \quad \sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -\frac{1}{4}\,{s_3}^2\left(4\,s_4s_1 - {s_2}^2\right)$

$\det(g_O) = -{s_3}^2 s_1$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1 + B\left(x^4\right), x^2 = x^2, x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^4\right)]$$
$$\Phi_3 = [x^1 = x^1\xi_1, x^2 = x^2\xi_2, x^3 = \xi_2 x^3, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^4\right), 0, s_3\left(x^4\right), e], [e^2 = 1]]$	timelike and spacelike orbits
$[[0, 2\,e, s_3\left(x^4\right), 0], [e^2 = 1]]$	null orbits, non-Lorentzian signature, isometry dim 6

11. Petrov Reference: [[32, 11, 0], [32, 27, 0]]

[4, 3, 12]

1. Reference : B(4,4), Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	$-\frac{1}{2}Y_2+\frac{1}{2}Y_3$	$\frac{1}{2}Y_2-\frac{1}{2}Y_3$	.
Y_2		.	.	$-Y_3$
Y_3			.	$-Y_2$
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	e_1	.	.
e_2		.	.	.
e_3			.	e_3
e_4				.

3. Isomorphisms:

$[X_1 \to -Y_2 + Y_3, X_2 \to Y_2 + Y_3, X_3 \to Y_1, X_4 \to Y_1 - Y_4]$
$[X_1 \to e_1, X_2 \to -e_3, X_3 \to e_2, X_4 \to e_4]$

4. Isotropy: F13 $[e_2 - e_4]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2} \quad X_3 = -\partial_{x^1} + x^2\,\partial_{x^2}$$
$$X_2 = \partial_{x^3} \quad X_4 = -\partial_{x^1} + x^3\,\partial_{x^3}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1 \qquad \sigma^3 = \mathrm{e}^{x^1}\,dx^2\,dx^3$$
$$\sigma^2 = \frac{1}{2}\,dx^1\,dx^4 \quad \sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -\frac{1}{4}\,{s_3}^2\mathrm{e}^{2\,x^1}\left(4\,s_4 s_1 - {s_2}^2\right)$

$\det(g_O) = -s_1{s_3}^2\mathrm{e}^{2\,x^1}$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1 + B\left(x^4\right), x^2 = x^2, x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^4\right)]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^4\right), 0, s_3\left(x^4\right), e], [e^2 = 1]]$ timelike and spacelike orbits
$[[0, 2\,e, s_3\left(x^4\right), 0], [e^2 = 1]]$ null orbits, non-Lorentzian signature

11. Petrov Reference: [[32, 5, 0], [32, 21, 0]]

A.2.3.3 $F14$

[4, 3, 13]

1. Reference : N(4,1), Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	$-Y_2$	$-Y_3$	$-Y_2$
Y_2		.	.	Y_3
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	e_1
e_2		.	e_1	e_2
e_3			.	.
e_4				.

3. Isomorphisms:

$[X_1 \to Y_3, X_2 \to Y_2, X_3 \to -Y_2 + Y_4, X_4 \to Y_1 + Y_3]$
$[X_1 \to e_1, X_2 \to -e_1 + e_2, X_3 \to e_1 + e_3, X_4 \to -2\,e_1 + e_2 + e_4]$

4. Isotropy: F14 $[\,e_2 + e_3]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2} \quad X_3 = x^3\,\partial_{x^2} - x^1\,\partial_{x^3}$$
$$X_2 = \partial_{x^3} \quad X_4 = x^1\,\partial_{x^1} + x^2\,\partial_{x^2} + x^3\,\partial_{x^3}$$

6. Base Point: $[1, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = {x^1}^{-2}\,dx^1\,dx^1 \qquad \sigma^3 = \frac{1}{2}\,{x^1}^{-1}\,dx^1\,dx^4$$
$$\sigma^2 = {x^1}^{-2}\,dx^1\,dx^2 + {x^1}^{-2}\,dx^3\,dx^3 \qquad \sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -\frac{{s_2}^3 s_4}{x^{16}}$

$\det(g_O) = -\frac{{s_2}^3}{x^{16}}$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1, x^2 = x^2 + B\left(x^4\right)x^1, x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^4\right)]$$
$$\Phi_3 = [x^1 = \frac{x^1}{\xi_1}, x^2 = x^2\xi_1, x^3 = x^3, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^4\right), s_2\left(x^4\right), 0, e], [e^2 = 1]]$

11. Petrov Reference: [[32, 14, 1]]

[4, 3, 14]

1. Reference : N(4,2), Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	$-2\,Y_2+Y_4$	$-2\,Y_3$	$-Y_2$
Y_2		.	.	Y_3
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	$2\,e_1$
e_2		.	e_1	e_2
e_3			.	e_2+e_3
e_4				.

3. Isomorphisms:

$[X_1 \to Y_3, X_2 \to Y_2 - Y_4, X_3 \to Y_4, X_4 \to Y_1 + Y_3]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4]$

4. Isotropy: F14 $[\,e_3]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2} \quad X_3 = x^3\,\partial_{x^2} - x^1\,\partial_{x^3}$$
$$X_2 = \partial_{x^3} \quad X_4 = \partial_{x^1} + 2\,x^2\,\partial_{x^2} + x^3\,\partial_{x^3}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1 \qquad \sigma^3 = \frac{1}{2}\,dx^1\,dx^4$$
$$\sigma^2 = \mathrm{e}^{-2\,x^1}\,dx^1\,dx^2 + \mathrm{e}^{-2\,x^1}\,dx^3\,dx^3 \qquad \sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -{s_2}^3\mathrm{e}^{-6\,x^1}s_4$

$\det(g_O) = -{s_2}^3\mathrm{e}^{-6\,x^1}$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1, x^2 = x^2 + B\left(x^4\right)\mathrm{e}^{2\,x^1}, x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^4\right)]$$
$$\Phi_3 = [x^1 = x^1, x^2 = x^2\mathrm{e}^{\xi_1}, x^3 = x^3\mathrm{e}^{\frac{1}{2}\,\xi_1}, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^4\right), s_2\left(x^4\right), 0, e], [e^2 = 1]]$

11. Petrov Reference: [[32, 15, 0]]

[4, 3, 15]

1. Reference : N(4,3), Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	$-\frac{(1+a)Y_2}{a-1}+\frac{a\,Y_4}{(a-1)^2}$	$-\frac{(1+a)Y_3}{a-1}$	$-Y_2$
Y_2		.	.	Y_3
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	$(1+a)e_1$
e_2		.	e_1	e_2
e_3			.	$a\,e_3$
e_4				.

3. Isomorphisms:

$[X_1 \to -Y_3, X_2 \to -Y_2 + \frac{a}{a-1}\,Y_4, X_3 \to Y_2 - (a-1)^{-1}\,Y_4, X_4 \to (a-1)\,Y_1]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4]$

4. Isotropy: F14 $[e_2 + e_3]$

5. Vector Fields Γ: $[a \neq 1]$

$$X_1 = \partial_{x^2} \quad X_3 = x^3\,\partial_{x^2} - x^1\,\partial_{x^3}$$
$$X_2 = \partial_{x^3} \quad X_4 = (-(1+a)\,x^1 + 2\,x^1)\,\partial_{x^1} + (1+a)\,x^2\,\partial_{x^2} + x^3\,\partial_{x^3}$$

6. Base Point: $[1, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = {x^1}^{-2}\,dx^1\,dx^1 \qquad \sigma^3 = \frac{1}{2}\,{x^1}^{-1}\,dx^1\,dx^4$$
$$\sigma^2 = {x^1}^{2\,(a-1)^{-1}}\,dx^1\,dx^2 + {x^1}^{2\,(a-1)^{-1}}\,dx^3\,dx^3 \qquad \sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -{s_2}^3{x^1}^{6\,(a-1)^{-1}}\,s_4$

$\det(g_O) = -{s_2}^3{x^1}^{6\,(a-1)^{-1}}$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1, x^2 = x^2 + B\left(x^4\right)\left(x^1\right)^{-\frac{1+a}{a-1}}, x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^4\right)]$$
$$\Phi_3 = [x^1 = \frac{x^1}{\xi_1}, x^2 = x^2\xi_1, x^3 = x^3, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^4\right), s_2\left(x^4\right), 0, e], [e^2 = 1]]$

11. Petrov Reference: [[32, 14, 0]]

[4, 3, 16]

1. Reference : N(4,4), Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	$-2\,a\,Y_2+\left(a^2+1\right)Y_4$	$-2\,a\,Y_3$	$-Y_2$
Y_2		.	.	Y_3
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	$2\,a\,e_1$
e_2		.	e_1	$a\,e_2-e_3$
e_3			.	$a\,e_3+e_2$
e_4				.

3. Isomorphisms:

$[X_1 \to Y_3, X_2 \to -Y_4, X_3 \to Y_2 - a\,Y_4, X_4 \to Y_1]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4]$

4. Isotropy: F14 $[\,e_2]$

5. Vector Fields Γ: $[a \neq 0]$

$$X_1 = \partial_{x^3}$$
$$X_2 = -\frac{\cos\left(x^1\right)}{\sin\left(x^1\right)}\,\partial_{x^2} - \frac{1}{2}\,\frac{x^2\left(2\,a\,\sin\left(x^1\right)+\cos\left(x^1\right)\right)}{a\,\sin\left(x^1\right)}\,\partial_{x^3}$$
$$X_3 = \partial_{x^2} + \frac{1}{2}\,\frac{x^2}{a}\,\partial_{x^3}$$
$$X_4 = \partial_{x^1} + \frac{x^2\left(a\,\sin\left(x^1\right)-\cos\left(x^1\right)\right)}{\sin\left(x^1\right)}\,\partial_{x^2} + \frac{1}{2}\,\frac{4\,\sin\left(x^1\right)a^2x^3 - \sin\left(x^1\right)a\,x^4 - \cos\left(x^1\right)x^4}{a\,\sin\left(x^1\right)}\,\partial_{x^3}$$

6. Base Point: $[\pi/2, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1$$
$$\sigma^2 = -\frac{1}{2}\,\frac{x^2\,\mathrm{e}^{-2\,a\,x^1}}{a}\,dx^1\,dx^2 + \mathrm{e}^{-2\,a\,x^1}\,dx^1\,dx^3 - \frac{1}{2}\,\mathrm{e}^{-2\,a\,x^1}\left(\cos\left(2\,x^1\right)-1\right)\,dx^2\,dx^2$$
$$\sigma^3 = \frac{1}{2}\,dx^1\,dx^4$$
$$\sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -{s_2}^3\mathrm{e}^{-6\,a\,x^1}\,s_4\left(\sin\left(x^1\right)\right)^2$

$\det(g_O) = -{s_2}^3\mathrm{e}^{-6\,a\,x^1}\left(\sin\left(x^1\right)\right)^2$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1, x^2 = x^2, x^3 = x^3 + B\left(x^4\right)\mathrm{e}^{2\,a\,x^1}, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^4\right)]$$
$$\Phi_3 = [x^1 = x^1, x^2 = x^2\mathrm{e}^{\frac{1}{2}\,\xi_1}, x^3 = x^3\mathrm{e}^{\xi_1}, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^4\right), s_2\left(x^4\right), 0, e], [e^2 = 1]]$

11. Petrov Reference: [[32, 16, 0]]

[4, 3, 17]

1. Reference : N(4,5), Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	Y_4	.	$-Y_2$
Y_2		.	.	Y_3
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	.
e_2		.	e_1	$-e_3$
e_3			.	e_2
e_4				.

3. Isomorphisms:

$[X_1 \to Y_3, X_2 \to Y_2, X_3 \to Y_4, X_4 \to -Y_1 + Y_3]$
$[X_1 \to e_1, X_2 \to -e_1 - e_3, X_3 \to e_2, X_4 \to e_3 - e_4]$

4. Isotropy: F14 $[e_2]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2} \quad X_3 = x^3\,\partial_{x^2} - x^1\,\partial_{x^3}$$
$$X_2 = \partial_{x^3} \quad X_4 = (-\left(x^1\right)^2 - 1)\,\partial_{x^1} + \frac{1}{2}\,x^6\,\partial_{x^2} - x^1x^3\,\partial_{x^3}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = \left(x^{1\,2} + 1\right)^{-2} dx^1\,dx^1$$
$$\sigma^2 = \left(x^{1\,2} + 1\right)^{-1} dx^1\,dx^2 + \left(x^{1\,2} + 1\right)^{-1} dx^3\,dx^3$$
$$\sigma^3 = \frac{1}{2}\left(x^{1\,2} + 1\right)^{-1} dx^1\,dx^4$$
$$\sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -\frac{{s_2}^3 s_4}{\left(x^{1\,2}+1\right)^3}$

$\det(g_O) = -\frac{{s_2}^3}{\left(x^{1\,2}+1\right)^3}$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1, x^2 = x^2 + B\left(x^4\right), x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^4\right)]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^4\right), s_2\left(x^4\right), 0, e], [e^2 = 1]]$

11. Petrov Reference: [[32, 16, 1]]

[4, 3, 18]

1. Reference : N(4,6), Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	$-\frac{1}{4}Y_4$	.	$-Y_2$
Y_2		.	.	Y_3
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	.
e_2		.	e_1	e_2
e_3			.	$-e_3$
e_4				.

3. Isomorphisms:

$[X_1 \to Y_3, X_2 \to Y_2 + \frac{1}{2}Y_4, X_3 \to -Y_2 + \frac{1}{2}Y_4, X_4 \to 2Y_1 + Y_3]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4]$

4. Isotropy: F14 $[\frac{1}{2}e_2 + \frac{1}{2}e_3]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2} \quad X_3 = x^3\,\partial_{x^2} - x^1\,\partial_{x^3}$$
$$X_2 = \partial_{x^3} \quad X_4 = 2\,x^1\,\partial_{x^1} + x^3\,\partial_{x^3}$$

6. Base Point: $[1, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = {x^1}^{-2}\,dx^1\,dx^1$$
$$\sigma^2 = {x^1}^{-1}\,dx^1\,dx^2 + {x^1}^{-1}\,dx^3\,dx^3$$
$$\sigma^3 = \frac{1}{2}\,{x^1}^{-1}\,dx^1\,dx^4$$
$$\sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -\frac{{s_2}^3 s_4}{{x^1}^3}$

$\det(g_O) = -\frac{{s_2}^3}{{x^1}^3}$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1, x^2 = x^2 + B\left(x^4\right), x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^4\right)]$$
$$\Phi_3 = [x^1 = x^1 \mathrm{e}^{-\xi_2}, x^2 = \frac{1}{2}\,\ln\left(x^1\right)\xi_1 - \frac{1}{2}\,\xi_1\xi_2 + x^2\mathrm{e}^{\xi_2}, x^3 = x^3, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^4\right), s_2\left(x^4\right), 0, e], [e^2 = 1]]$

11. Petrov Reference: [[32, 14, 2]]

[4, 3, 19]

1. Reference : N(4,7), Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	$-Y_2$
Y_2		.	.	Y_3
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	.
e_2		.	.	e_1
e_3			.	e_2
e_4				.

3. Isomorphisms:

$[X_1 \to -Y_1 + Y_3, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_2 - e_3, X_2 \to -e_1 + e_2, X_3 \to -e_1, X_4 \to -e_4]$

4. Isotropy: F14 $[e_4]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2} \quad X_3 = -\partial_{x^1}$$
$$X_2 = \partial_{x^3} \quad X_4 = -x^3\,\partial_{x^1} + x^2\,\partial_{x^3}$$

6. Base Point: $[1, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = dx^1\,dx^2 + dx^3\,dx^3 \quad \sigma^3 = \frac{1}{2}\,dx^2\,dx^4$$
$$\sigma^2 = dx^2\,dx^2 \qquad \sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -{s_1}^3 s_4$

$\det(g_O) = -{s_1}^3$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1 + B\left(x^4\right), x^2 = x^2, x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^4\right)]$$
$$\Phi_3 = [x^1 = -\xi_1 x^2 \xi_3 + \frac{x^1 {\xi_2}^2}{\xi_3}, x^2 = x^2 \xi_3, x^3 = \xi_2 x^3, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1\left(x^4\right), s_2\left(x^4\right), 0, e], [e^2 = 1]]$

11. Petrov Reference: [[32, 12, 0]]

[4, 3, 20]

1. Reference : N(4,8), Bowers

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	$Y_1 - Y_3$	$-Y_2$	$-Y_2$
Y_2		.	Y_3	Y_3
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	e_1	$-2\,e_2$	.
e_2		.	e_3	.
e_3			.	.
e_4				.

3. Isomorphisms:

$[X_1 \to Y_3, X_2 \to -Y_2, X_3 \to -2\,Y_1 + Y_3, X_4 \to Y_3 - Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to -e_3, X_4 \to -e_4]$

4. Isotropy: F14 $[\,e_1 + e_4]$

5. Vector Fields Γ:

$$X_1 = \mathrm{e}^{-x^3}\,\partial_{x^1} - \mathrm{e}^{-x^3} x^4\,\partial_{x^2} - 2\,\mathrm{e}^{-x^3} x^2\,\partial_{x^3} \qquad X_3 = \mathrm{e}^{x^3}\,\partial_{x^2}$$
$$X_2 = \partial_{x^3} \qquad X_4 = \partial_{x^1}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = dx^1\,dx^2 - x^2\,dx^1\,dx^3 - \frac{1}{2}\,dx^3\,dx^3$$
$$\sigma^2 = dx^2\,dx^2 - x^2\,dx^2\,dx^3 + {x^2}^2\,dx^3\,dx^3$$
$$\sigma^3 = -\frac{1}{2}\,dx^2\,dx^4 + \frac{x^2}{2}\,dx^3\,dx^4$$
$$\sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = \frac{1}{2}\,{s_1}^3 s_4$

$\det(g_O) = \frac{1}{2}\,{s_1}^3$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1 + B\left(x^4\right), x^2 = x^2, x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^4\right)]$$
$$\Phi_3 = [x^1 = x^1 \mathrm{e}^{\xi_1}, x^2 = x^2 \mathrm{e}^{-\xi_1}, x^3 = x^3 - \xi_1, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$$[[s_1\left(x^4\right), s_2\left(x^4\right), 0, e], [e^2 = 1]]$$

11. Petrov Reference: [[32, 8, 0]]

A.2.4 G_4 on V_4

[4, 4, 1]

1. Reference : so(3)+n(1,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	Y_3	$-Y_2$	.
Y_2		.	Y_1	.
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	e_3	$-e_2$	.
e_2		.	e_1	.
e_3			.	.
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^2}$$

$$X_2 = \cos\left(x^2\right)\,\partial_{x^1} - \frac{\cos\left(x^1\right)\sin\left(x^2\right)}{\sin\left(x^1\right)}\,\partial_{x^2} + \frac{\sin\left(x^2\right)}{\sin\left(x^1\right)}\,\partial_{x^3}$$

$$X_3 = -\sin\left(x^2\right)\,\partial_{x^1} - \frac{\cos\left(x^1\right)\cos\left(x^2\right)}{\sin\left(x^1\right)}\,\partial_{x^2} + \frac{\cos\left(x^2\right)}{\sin\left(x^1\right)}\,\partial_{x^3}$$

$$X_4 = \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = -2\,\sin\left(x^3\right)\cos\left(x^3\right)\,dx^1\,dx^1 + \cos\left(2\,x^3\right)\sin\left(x^1\right)\,dx^1\,dx^2 + 2\,\left(\sin\left(x^1\right)\right)^2\cos\left(x^3\right)\sin\left(x^3\right)\,dx^2\,dx^2$$

$$\sigma^2 = dx^1\,dx^1 + \left(\sin\left(x^1\right)\right)^2\,dx^2\,dx^2$$

$$\sigma^3 = dx^1\,dx^1 + dx^2\,dx^2 + \cos\left(x^1\right)\,dx^2\,dx^3 + dx^3\,dx^3$$

$$\sigma^4 = -\cos\left(2\,x^3\right)\,dx^1\,dx^1 - 2\,\sin\left(x^3\right)\cos\left(x^3\right)\sin\left(x^1\right)\,dx^1\,dx^2 + \cos\left(2\,x^3\right)\left(\sin\left(x^1\right)\right)^2\,dx^2\,dx^2$$

$$\sigma^5 = -\frac{1}{2}\,\cos\left(x^1\right)\sin\left(x^3\right)\,dx^1\,dx^2 - \frac{1}{2}\,\sin\left(x^3\right)\,dx^1\,dx^3 + \cos\left(x^1\right)\cos\left(x^3\right)\sin\left(x^1\right)\,dx^2\,dx^2 + \frac{1}{2}\,\cos\left(x^3\right)\sin\left(x^1\right)\,dx^2\,dx^3$$

$$\sigma^6 = \frac{1}{2}\,\cos\left(x^1\right)\cos\left(x^3\right)\,dx^1\,dx^2 + \frac{1}{2}\,\cos\left(x^3\right)\,dx^1\,dx^3 + \cos\left(x^1\right)\sin\left(x^3\right)\sin\left(x^1\right)\,dx^2\,dx^2 + \frac{1}{2}\,\sin\left(x^3\right)\sin\left(x^1\right)\,dx^2\,dx^3$$

$$\sigma^7 = -\frac{1}{2}\,\sin\left(x^3\right)\,dx^1\,dx^4 + \frac{1}{2}\,\cos\left(x^3\right)\sin\left(x^1\right)\,dx^2\,dx^4$$

$$\sigma^8 = \frac{1}{2}\,\cos\left(x^3\right)\,dx^1\,dx^4 + \frac{1}{2}\,\sin\left(x^3\right)\sin\left(x^1\right)\,dx^2\,dx^4$$

$$\sigma^9 = \frac{1}{2}\,\cos\left(x^1\right)\,dx^2\,dx^4 + \frac{1}{2}\,dx^3\,dx^4$$

$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[32, 47, 0]]

[4, 4, 2]

1. Reference : so(2,1)+n(1,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	Y_1	$2\,Y_2$	.
Y_2		.	Y_3	.
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	e_1	$2\,e_2$	.
e_2		.	e_3	.
e_3			.	.
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4]$

4. Vector Fields Γ:

$$X_1 = \mathrm{e}^{-x^3}\,\partial_{x^1} - \mathrm{e}^{-x^3} x^4\,\partial_{x^2} - 2\,\mathrm{e}^{-x^3} x^2\,\partial_{x^3}$$
$$X_2 = \partial_{x^3}$$
$$X_3 = \mathrm{e}^{x^3}\,\partial_{x^2}$$
$$X_4 = \partial_{x^1} - \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\begin{aligned}
\sigma^1 = {}& \frac{1}{48}\,dx^1\,dx^1 \\
&+ (\frac{1}{48}\,{x^1}^2 + \frac{1}{24}\,x^1\,x^4 + \frac{1}{48}\,{x^4}^2)\,dx^1\,dx^2 \\
&+ (-\frac{1}{48}\,x^2\,{x^1}^2 - \frac{1}{24}\,x^2\,x^1\,x^4 - \frac{1}{48}\,x^2\,{x^4}^2 + \frac{x^1}{48} + \frac{x^4}{48})\,dx^1\,dx^3 \\
&+ (\frac{1}{48}\,{x^1}^4 + \frac{1}{12}\,x^4\,{x^1}^3 + \frac{1}{8}\,{x^4}^2{x^1}^2 + \frac{1}{12}\,x^1\,{x^4}^3 + \frac{1}{48}\,{x^4}^4)\,dx^2\,dx^2 \\
&+ (\frac{1}{48}\,{x^1}^3 - \frac{1}{48}\,{x^1}^4x^2 - \frac{1}{48}\,x^2\,{x^4}^4 + \frac{1}{48}\,{x^4}^3 - \frac{1}{12}\,{x^1}^3x^2\,x^4 - \frac{1}{8}\,{x^1}^2x^2\,{x^4}^2 \\
&- \frac{1}{12}\,x^1\,x^2\,{x^4}^3 + \frac{1}{16}\,{x^1}^2x^4 + \frac{1}{16}\,x^1\,{x^4}^2)\,dx^2\,dx^3 \\
&+ (\frac{1}{12}\,{x^2}^2x^1\,{x^4}^3 + \frac{1}{8}\,{x^2}^2{x^4}^2{x^1}^2 + \frac{1}{12}\,{x^2}^2x^4\,{x^1}^3 - \frac{1}{8}\,x^2\,{x^1}^2x^4 - \frac{1}{8}\,x^2\,x^1\,{x^4}^2 \\
&+ \frac{1}{48}\,{x^4}^2 + \frac{1}{48}\,{x^1}^2 + \frac{1}{48}\,{x^2}^2{x^4}^4 + \frac{1}{48}\,{x^1}^4{x^2}^2 - \frac{1}{24}\,{x^1}^3x^2 \\
&- \frac{1}{24}\,x^2\,{x^4}^3 + \frac{1}{24}\,x^1\,x^4)\,dx^3\,dx^3 \\
\sigma^2 = {}& \frac{1}{8}\,dx^1\,dx^2 \\
&- x^2/8\,dx^1\,dx^3 \\
&+ (\frac{1}{4}\,{x^1}^2 + \frac{1}{2}\,x^1\,x^4 + \frac{1}{4}\,{x^4}^2)\,dx^2\,dx^2 \\
&+ (-\frac{1}{4}\,x^2\,{x^1}^2 - \frac{1}{4}\,x^2\,{x^4}^2 + x^4/8 + x^1/8 - \frac{1}{2}\,x^2\,x^1\,x^4)\,dx^2\,dx^3 \\
&+ (\frac{1}{2}\,{x^2}^2x^4\,x^1 + \frac{1}{4}\,{x^2}^2{x^4}^2 + \frac{1}{4}\,{x^2}^2{x^1}^2 - \frac{1}{4}\,x^2\,x^1 - \frac{1}{4}\,x^2\,x^4)\,dx^3\,dx^3 \\
\sigma^3 = {}& (x^1/24 + x^4/24)\,dx^1\,dx^2
\end{aligned}$$

$$
\begin{aligned}
&+ (-\frac{1}{24}\,x^2\,x^1 - \frac{1}{24}\,x^2\,x^4 + \frac{1}{48})\,dx^1\,dx^3\\
&+ (\frac{1}{12}\,{x^1}^3 + \frac{1}{4}\,{x^1}^2 x^4 + \frac{1}{4}\,x^1\,{x^4}^2 + \frac{1}{12}\,{x^4}^3)\,dx^2\,dx^2\\
&+ (-\frac{1}{12}\,{x^1}^3 x^2 - \frac{1}{12}\,x^2\,{x^4}^3 + \frac{1}{16}\,{x^4}^2 + \frac{1}{16}\,{x^1}^2 - \frac{1}{4}\,x^2\,{x^1}^2 x^4 - \frac{1}{4}\,x^2\,x^1\,{x^4}^2\\
&+ \frac{1}{8}\,x^1\,x^4)\,dx^2\,dx^3\\
&+ (\frac{1}{4}\,{x^2}^2 {x^4}^2 x^1 + \frac{1}{4}\,{x^2}^2 x^4\,{x^1}^2 - \frac{1}{4}\,x^2\,x^1\,x^4 + \frac{x^4}{24} + \frac{x^1}{24} + \frac{1}{12}\,{x^2}^2 {x^4}^3 + \frac{1}{12}\,{x^2}^2 {x^1}^3\\
&- \frac{1}{8}\,x^2\,{x^1}^2 - \frac{1}{8}\,x^2\,{x^4}^2)\,dx^3\,dx^3\\
\sigma^4 &= -2\,dx^1\,dx^2\\
&+ 2\,x^2\,dx^1\,dx^3\\
&+ \,dx^3\,dx^3\\
\sigma^5 &= -\frac{1}{4}\,dx^1\,dx^4\\
&+ (-\frac{1}{4}\,{x^1}^2 - \frac{1}{2}\,x^1\,x^4 - \frac{1}{4}\,{x^4}^2)\,dx^2\,dx^4\\
&+ (\frac{1}{4}\,x^2\,{x^1}^2 + \frac{1}{2}\,x^2\,x^1\,x^4 + \frac{1}{4}\,x^2\,{x^4}^2 - x^1/4 - x^4/4)\,dx^3\,dx^4\\
\sigma^6 &= \frac{1}{2}\,dx^2\,dx^2\\
&- x^2/2\,dx^2\,dx^3\\
&+ \frac{1}{2}\,{x^2}^2\,dx^3\,dx^3\\
\sigma^7 &= (x^1/2 + x^4/2)\,dx^2\,dx^2\\
&+ (\frac{1}{8} - \frac{1}{2}\,x^2\,x^1 - \frac{1}{2}\,x^2\,x^4)\,dx^2\,dx^3\\
&+ (-x^2/4 + \frac{1}{2}\,{x^2}^2 x^4 + \frac{1}{2}\,{x^2}^2 x^1)\,dx^3\,dx^3\\
\sigma^8 &= -\frac{1}{2}\,dx^2\,dx^4\\
&+ x^2/2\,dx^3\,dx^4\\
\sigma^9 &= (-x^1/2 - x^4/2)\,dx^2\,dx^4\\
&+ (\frac{1}{2}\,x^2\,x^1 + \frac{1}{2}\,x^2\,x^4 - \frac{1}{4})\,dx^3\,dx^4\\
\sigma^{10} &= \,dx^4\,dx^4
\end{aligned}
$$

6. Petrov Reference: [[32, 46, 0], [32, 46, 1]]

[4, 4, 3]

1. Reference : s(3,3)+n(1,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	αY_1+Y_2
Y_2		.	.	αY_2-Y_1
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	$-\alpha e_1+e_2$	.
e_2		.	$-\alpha e_2-e_1$	.
e_3			.	.
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_2, X_2 \to e_1, X_3 \to e_4, X_4 \to -e_3]$

4. Vector Fields Γ: $[0 \leq \alpha]$

$$
\begin{aligned}
X_1 &= \partial_{x^2} \\
X_2 &= \partial_{x^3} \\
X_3 &= -\partial_{x^1} \\
X_4 &= (\alpha x^2 - x^3)\,\partial_{x^2} + (\alpha x^3 + x^2)\,\partial_{x^3} + \partial_{x^4}
\end{aligned}
$$

5. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$
\begin{aligned}
\sigma^1 &= dx^1\,dx^1 \\
\sigma^2 &= -\frac{1}{2}\,\mathrm{e}^{-\alpha x^4} \sin\left(x^4\right)\,dx^1\,dx^2 + \frac{1}{2}\,\mathrm{e}^{-\alpha x^4} \cos\left(x^4\right)\,dx^1\,dx^3 \\
\sigma^3 &= \frac{1}{2}\,\mathrm{e}^{-\alpha x^4} \cos\left(x^4\right)\,dx^1\,dx^2 + \frac{1}{2}\,\mathrm{e}^{-\alpha x^4} \sin\left(x^4\right)\,dx^1\,dx^3 \\
\sigma^4 &= \frac{1}{2}\,dx^1\,dx^4 \\
\sigma^5 &= -\mathrm{e}^{-2\alpha x^4} \cos\left(2x^4\right)\,dx^2\,dx^2 - \mathrm{e}^{-2\alpha x^4} \sin\left(2x^4\right)\,dx^2\,dx^3 + \mathrm{e}^{-2\alpha x^4} \cos\left(2x^4\right)\,dx^3\,dx^3 \\
\sigma^6 &= -\mathrm{e}^{-2\alpha x^4} \sin\left(2x^4\right)\,dx^2\,dx^2 + \mathrm{e}^{-2\alpha x^4} \cos\left(2x^4\right)\,dx^2\,dx^3 + \mathrm{e}^{-2\alpha x^4} \sin\left(2x^4\right)\,dx^3\,dx^3 \\
\sigma^7 &= \mathrm{e}^{-2\alpha x^4}\,dx^2\,dx^2 + \mathrm{e}^{-2\alpha x^4}\,dx^3\,dx^3 \\
\sigma^8 &= -\frac{1}{2}\,\mathrm{e}^{-\alpha x^4} \sin\left(x^4\right)\,dx^2\,dx^4 + \frac{1}{2}\,\mathrm{e}^{-\alpha x^4} \cos\left(x^4\right)\,dx^3\,dx^4 \\
\sigma^9 &= \frac{1}{2}\,\mathrm{e}^{-\alpha x^4} \cos\left(x^4\right)\,dx^2\,dx^4 + \frac{1}{2}\,\mathrm{e}^{-\alpha x^4} \sin\left(x^4\right)\,dx^3\,dx^4 \\
\sigma^{10} &= dx^4\,dx^4
\end{aligned}
$$

6. Petrov Reference: [[32, 42, 1]]

[4, 4, 4]

1. Reference : s(3,2)+n(1,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	$-Y_1$	.
Y_2		.	$-Y_1-Y_2$	.
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	$-e_1$	.
e_2		.	$-e_1-e_2$	.
e_3			.	.
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^1}$$
$$X_2 = \partial_{x^1} + \partial_{x^2}$$
$$X_3 = (-x^2 - x^1)\,\partial_{x^1} - x^2\,\partial_{x^2} + \partial_{x^3}$$
$$X_4 = \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = \mathrm{e}^{2\,x^3}\,dx^1\,dx^1 + \mathrm{e}^{2\,x^3}x^3\,dx^1\,dx^2 + \mathrm{e}^{2\,x^3}{x^3}^2\,dx^2\,dx^2$$
$$\sigma^2 = \frac{1}{2}\,\mathrm{e}^{2\,x^3}\,dx^1\,dx^2 + \mathrm{e}^{2\,x^3}x^3\,dx^2\,dx^2$$
$$\sigma^3 = \frac{1}{2}\,\mathrm{e}^{x^3}\,dx^1\,dx^3 + \frac{1}{2}\,\mathrm{e}^{x^3}x^3\,dx^2\,dx^3$$
$$\sigma^4 = \frac{1}{2}\,\mathrm{e}^{x^3}\,dx^1\,dx^4 + \frac{1}{2}\,\mathrm{e}^{x^3}x^3\,dx^2\,dx^4$$
$$\sigma^5 = \mathrm{e}^{2\,x^3}\,dx^2\,dx^2$$
$$\sigma^6 = \frac{1}{2}\,\mathrm{e}^{x^3}\,dx^2\,dx^3$$
$$\sigma^7 = \frac{1}{2}\,\mathrm{e}^{x^3}\,dx^2\,dx^4$$
$$\sigma^8 = dx^3\,dx^3$$
$$\sigma^9 = \frac{1}{2}\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[32, 43, 0], [32, 44, 0]]

[4, 4, 5]

1. Reference : s(3,1)+n(1,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	$-Y_1$	.
Y_2		.	$-aY_2$	.
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	$-e_1$	.
e_2		.	$-a\,e_2$	.
e_3			.	.
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4]$

4. Vector Fields Γ: $[0 < \sqrt{a^2}, \sqrt{a^2} \leq 1]$

$$X_1 = \partial_{x^1}$$
$$X_2 = \partial_{x^2}$$
$$X_3 = -x^1\,\partial_{x^1} - ax^2\,\partial_{x^2} + \partial_{x^3}$$
$$X_4 = \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = \mathrm{e}^{2\,x^3}\,dx^1\,dx^1$$
$$\sigma^2 = \frac{1}{2}\,\mathrm{e}^{(a+1)x^3}\,dx^1\,dx^2$$
$$\sigma^3 = \frac{1}{2}\,\mathrm{e}^{x^3}\,dx^1\,dx^3$$
$$\sigma^4 = \frac{1}{2}\,\mathrm{e}^{x^3}\,dx^1\,dx^4$$
$$\sigma^5 = \mathrm{e}^{2\,ax^3}\,dx^2\,dx^2$$
$$\sigma^6 = \frac{1}{2}\,\mathrm{e}^{ax^3}\,dx^2\,dx^3$$
$$\sigma^7 = \frac{1}{2}\,\mathrm{e}^{ax^3}\,dx^2\,dx^4$$
$$\sigma^8 = dx^3\,dx^3$$
$$\sigma^9 = \frac{1}{2}\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[32, 41, 0]]

[4, 4, 6]

1. Reference : s(3,1)+n(1,1) a=1, Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	.
Y_2		.	.	$-Y_2$
Y_3			.	$-Y_3$
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	$-e_1$	.
e_2		.	$-e_2$	.
e_3			.	.
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_4, X_2 \to e_1, X_3 \to e_2, X_4 \to e_3]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^2}$$
$$X_2 = \partial_{x^3}$$
$$X_3 = -\partial_{x^1}$$
$$X_4 = -x^1\,\partial_{x^1} - x^3\,\partial_{x^3} + \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = \mathrm{e}^{2\,x^4}\,dx^1\,dx^1$$
$$\sigma^2 = \frac{1}{2}\,\mathrm{e}^{x^4}\,dx^1\,dx^2$$
$$\sigma^3 = \frac{1}{2}\,\mathrm{e}^{2\,x^4}\,dx^1\,dx^3$$
$$\sigma^4 = \frac{1}{2}\,\mathrm{e}^{x^4}\,dx^1\,dx^4$$
$$\sigma^5 = dx^2\,dx^2$$
$$\sigma^6 = \frac{1}{2}\,\mathrm{e}^{x^4}\,dx^2\,dx^3$$
$$\sigma^7 = \frac{1}{2}\,dx^2\,dx^4$$
$$\sigma^8 = \mathrm{e}^{2\,x^4}\,dx^3\,dx^3$$
$$\sigma^9 = \frac{1}{2}\,\mathrm{e}^{x^4}\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[32, 41, 2]]

[4, 4, 7]

1. Reference : s(4,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	Y_2
Y_2		.	.	.
Y_3			.	Y_3
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	.
e_2		.	.	$-e_1$
e_3			.	$-e_3$
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_1 + e_2, X_2 \to e_1, X_3 \to -e_3, X_4 \to 2\,e_1 + e_2 + e_3 - e_4]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^2}$$
$$X_2 = \partial_{x^3}$$
$$X_3 = -\partial_{x^1}$$
$$X_4 = x^1\,\partial_{x^1} + x^2\,\partial_{x^3} + \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = \mathrm{e}^{-2\,x^4}\,dx^1\,dx^1$$
$$\sigma^2 = \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^1\,dx^2$$
$$\sigma^3 = -\frac{1}{2}\,x^4\,\mathrm{e}^{-x^4}\,dx^1\,dx^2 + \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^1\,dx^3$$
$$\sigma^4 = \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^1\,dx^4$$
$$\sigma^5 = dx^2\,dx^2$$
$$\sigma^6 = -x^4\,dx^2\,dx^2 + \frac{1}{2}\,dx^2\,dx^3$$
$$\sigma^7 = {x^4}^2\,dx^2\,dx^2 - x^4\,dx^2\,dx^3 + dx^3\,dx^3$$
$$\sigma^8 = \frac{1}{2}\,dx^2\,dx^4$$
$$\sigma^9 = -\frac{x^4}{2}\,dx^2\,dx^4 + \frac{1}{2}\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[32, 43, 2]]

[4, 4, 8]

1. Reference : s(2,1)+s(2,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	$-Y_1$	.	.
Y_2		.	.	.
Y_3			.	$-Y_3$
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	$-e_1$	.	.
e_2		.	.	.
e_3			.	$-e_3$
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^1}$$
$$X_2 = -x^1\,\partial_{x^1} + \partial_{x^2}$$
$$X_3 = \partial_{x^3}$$
$$X_4 = -x^3\,\partial_{x^3} + \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = \mathrm{e}^{2\,x^2}\,dx^1\,dx^1$$
$$\sigma^2 = \frac{1}{2}\,\mathrm{e}^{x^2}\,dx^1\,dx^2$$
$$\sigma^3 = \frac{1}{2}\,\mathrm{e}^{x^2+x^4}\,dx^1\,dx^3$$
$$\sigma^4 = \frac{1}{2}\,\mathrm{e}^{x^2}\,dx^1\,dx^4$$
$$\sigma^5 = dx^2\,dx^2$$
$$\sigma^6 = \frac{1}{2}\,\mathrm{e}^{x^4}\,dx^2\,dx^3$$
$$\sigma^7 = \frac{1}{2}\,dx^2\,dx^4$$
$$\sigma^8 = \mathrm{e}^{2\,x^4}\,dx^3\,dx^3$$
$$\sigma^9 = \frac{1}{2}\,\mathrm{e}^{x^4}\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[32, 38, 0], [32, 39, 0]]

[4, 4, 9]

1. Reference : n(4,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	Y_2
Y_2		.	.	Y_3
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	.
e_2		.	.	e_1
e_3			.	e_2
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_3, X_2 \to e_2, X_3 \to e_1, X_4 \to e_4]$

4. Vector Fields Γ:

$$
\begin{aligned}
X_1 &= \partial_{x^2}\\
X_2 &= \partial_{x^3}\\
X_3 &= -\partial_{x^1}\\
X_4 &= -x^3\,\partial_{x^1} + x^2\,\partial_{x^3} + \partial_{x^4}
\end{aligned}
$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$
\begin{aligned}
\sigma^1 &= \frac{1}{2}\,dx^1\,dx^1 - \frac{1}{4}\,{x^4}^2\,dx^1\,dx^2 + \frac{1}{2}x^4\,dx^1\,dx^3 + \frac{1}{8}\,{x^4}^4\,dx^2\,dx^2\\
&\quad - \frac{1}{4}\,{x^4}^3\,dx^2\,dx^3 + \frac{1}{2}\,{x^4}^2\,dx^3\,dx^3\\
\sigma^2 &= \frac{1}{2}\,dx^1\,dx^2 - \frac{1}{2}\,{x^4}^2\,dx^2\,dx^2 + \frac{1}{2}x^4\,dx^2\,dx^3\\
\sigma^3 &= dx^1\,dx^2 + dx^3\,dx^3\\
\sigma^4 &= -\frac{1}{2}x^4\,dx^1\,dx^2 + \frac{1}{2}\,dx^1\,dx^3 + \frac{1}{2}\,{x^4}^3\,dx^2\,dx^2 - \frac{3}{4}\,{x^4}^2\,dx^2\,dx^3\\
&\quad + x^4\,dx^3\,dx^3\\
\sigma^5 &= \frac{1}{2}\,dx^1\,dx^4 - \frac{1}{4}\,{x^4}^2\,dx^2\,dx^4 + \frac{1}{2}x^4\,dx^3\,dx^4\\
\sigma^6 &= -x^4\,dx^2\,dx^2 + \frac{1}{2}\,dx^2\,dx^3\\
\sigma^7 &= dx^2\,dx^2\\
\sigma^8 &= \frac{1}{2}\,dx^2\,dx^4\\
\sigma^9 &= -\frac{1}{2}x^4\,dx^2\,dx^4 + \frac{1}{2}\,dx^3\,dx^4\\
\sigma^{10} &= dx^4\,dx^4
\end{aligned}
$$

6. Petrov Reference: [[32, 45, 0]]

[4, 4, 10]

1. Reference : s(4,3), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	Y_1
Y_2		.	.	$a\,Y_2$
Y_3			.	$b\,Y_3$
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-e_1$
e_2		.	.	$-a\,e_2$
e_3			.	$-b\,e_3$
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to -e_4]$

4. Vector Fields Γ: $[0 < \sqrt{b^2}, \sqrt{b^2} \le \sqrt{a^2}, a \ne -1, b \ne -1]$

$$X_1 = \partial_{x^2}$$
$$X_2 = \partial_{x^3}$$
$$X_3 = -\partial_{x^1}$$
$$X_4 = x^1\,b\,\partial_{x^1} + x^2\,\partial_{x^2} + x^3\,a\,\partial_{x^3} + \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = \frac{1}{2}\,dx^1\,dx^1 - \frac{1}{4}\,{x^4}^2\,dx^1\,dx^2 + \frac{1}{2}x^4\,dx^1\,dx^3 + \frac{1}{8}\,{x^4}^4\,dx^2\,dx^2 - \frac{1}{4}\,{x^4}^3\,dx^2\,dx^3 + \frac{1}{2}\,{x^4}^2\,dx^3\,dx^3$$
$$\sigma^2 = \frac{1}{2}\,dx^1\,dx^2 - \frac{1}{2}\,{x^4}^2\,dx^2\,dx^2 + \frac{1}{2}x^4\,dx^2\,dx^3$$
$$\sigma^3 = dx^1\,dx^2 + dx^3\,dx^3$$
$$\sigma^4 = -\frac{1}{2}x^4\,dx^1\,dx^2 + \frac{1}{2}\,dx^1\,dx^3 + \frac{1}{2}\,{x^4}^3\,dx^2\,dx^2 - \frac{3}{4}\,{x^4}^2\,dx^2\,dx^3 + x^4\,dx^3\,dx^3$$
$$\sigma^5 = \frac{1}{2}\,dx^1\,dx^4 - \frac{1}{4}\,{x^4}^2\,dx^2\,dx^4 + \frac{1}{2}x^4\,dx^3\,dx^4$$
$$\sigma^6 = -x^4\,dx^2\,dx^2 + \frac{1}{2}\,dx^2\,dx^3$$
$$\sigma^7 = dx^2\,dx^2$$
$$\sigma^8 = \frac{1}{2}\,dx^2\,dx^4$$
$$\sigma^9 = -\frac{1}{2}x^4\,dx^2\,dx^4 + \frac{1}{2}\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[32, 41, 1]]

[4, 4, 11]

1. Reference : s(4,3) (a,b)=(-1,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	Y_1
Y_2		.	.	$-Y_2$
Y_3			.	Y_3
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-e_1$
e_2		.	.	e_2
e_3			.	$-e_3$
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to -e_4]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^2}$$
$$X_2 = \partial_{x^3}$$
$$X_3 = -\partial_{x^1}$$
$$X_4 = x^1 \partial_{x^1} + x^2 \partial_{x^2} - x^3 \partial_{x^3} + \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = \mathrm{e}^{-2x^4} dx^1 dx^1$$
$$\sigma^2 = \frac{1}{2} \mathrm{e}^{-2x^4} dx^1 dx^2$$
$$\sigma^3 = \frac{1}{2} dx^1 dx^3$$
$$\sigma^4 = \frac{1}{2} \mathrm{e}^{-x^4} dx^1 dx^4$$
$$\sigma^5 = \mathrm{e}^{-2x^4} dx^2 dx^2$$
$$\sigma^6 = \frac{1}{2} dx^2 dx^3$$
$$\sigma^7 = \frac{1}{2} \mathrm{e}^{-x^4} dx^2 dx^4$$
$$\sigma^8 = \mathrm{e}^{2x^4} dx^3 dx^3$$
$$\sigma^9 = \frac{1}{2} \mathrm{e}^{x^4} dx^3 dx^4$$
$$\sigma^{10} = dx^4 dx^4$$

6. Petrov Reference: [[32, 41, 3]]

[4, 4, 12]

1. Reference : s(4,4), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	$\frac{Y_1}{a}+Y_2$
Y_2		.	.	$\frac{Y_2}{a}$
Y_3			.	Y_3
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-e_1$
e_2		.	.	$-e_1-e_2$
e_3			.	$-a\,e_3$
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_1 + e_2, X_2 \to a^{-1}\,e_1, X_3 \to e_3, X_4 \to -e_1 - e_2 - e_3 - a^{-1}\,e_4]$

4. Vector Fields Γ: $[a \neq 0]$

$$X_1 = \partial_{x^2}$$
$$X_2 = \partial_{x^3}$$
$$X_3 = -\partial_{x^1}$$
$$X_4 = x^1\,\partial_{x^1} + \frac{x^2}{a}\,\partial_{x^2} + (\frac{x^3}{a} + x^2)\,\partial_{x^3} + \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = \mathrm{e}^{-2\,x^4}\,dx^1\,dx^1$$
$$\sigma^2 = \frac{1}{2}\,\mathrm{e}^{-2\,x^4}\,dx^1\,dx^2$$
$$\sigma^3 = \frac{1}{2}\,dx^1\,dx^3$$
$$\sigma^4 = \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^1\,dx^4$$
$$\sigma^5 = \mathrm{e}^{-2\,x^4}\,dx^2\,dx^2$$
$$\sigma^6 = \frac{1}{2}\,dx^2\,dx^3$$
$$\sigma^7 = \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^2\,dx^4$$
$$\sigma^8 = \mathrm{e}^{2\,x^4}\,dx^3\,dx^3$$
$$\sigma^9 = \frac{1}{2}\,\mathrm{e}^{x^4}\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[32, 43, 1], [32, 44, 1]]

[4, 4, 13]

1. Reference : s(4,4) a=1, Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	Y_1+Y_2
Y_2		.	.	Y_2
Y_3			.	Y_3
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-e_1$
e_2		.	.	$-e_1-e_2$
e_3			.	$-e_3$
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_1 + e_2, X_2 \to e_1, X_3 \to e_3, X_4 \to -e_1 - e_2 - e_3 - e_4]$

4. Vector Fields Γ:

$$
\begin{aligned}
X_1 &= \partial_{x^2}\\
X_2 &= \partial_{x^3}\\
X_3 &= -\partial_{x^1}\\
X_4 &= x^1\,\partial_{x^1} + x^2\,\partial_{x^2} + (x^3 + x^2)\,\partial_{x^3} + \partial_{x^4}
\end{aligned}
$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$
\begin{aligned}
\sigma^1 &= \mathrm{e}^{-2\,x^4}\,dx^1\,dx^1\\
\sigma^2 &= \frac{1}{2}\,\mathrm{e}^{-2\,x^4}\,dx^1\,dx^2\\
\sigma^3 &= -\frac{1}{2}\,x^4\,\mathrm{e}^{-2\,x^4}\,dx^1\,dx^2 + \frac{1}{2}\,\mathrm{e}^{-2\,x^4}\,dx^1\,dx^3\\
\sigma^4 &= \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^1\,dx^4\\
\sigma^5 &= \mathrm{e}^{-2\,x^4}\,dx^2\,dx^2\\
\sigma^6 &= -x^4\,\mathrm{e}^{-2\,x^4}\,dx^2\,dx^2 + \frac{1}{2}\,\mathrm{e}^{-2\,x^4}\,dx^2\,dx^3\\
\sigma^7 &= {x^4}^2\mathrm{e}^{-2\,x^4}\,dx^2\,dx^2 - x^4\,\mathrm{e}^{-2\,x^4}\,dx^2\,dx^3 + \mathrm{e}^{-2\,x^4}\,dx^3\,dx^3\\
\sigma^8 &= \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^2\,dx^4\\
\sigma^9 &= -\frac{1}{2}\,x^4\,\mathrm{e}^{-x^4}\,dx^2\,dx^4 + \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^3\,dx^4\\
\sigma^{10} &= dx^4\,dx^4
\end{aligned}
$$

6. Petrov Reference: [[32, 43, 3]]

[4, 4, 14]

1. Reference : s(4,2), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	$Y_1{+}Y_2$
Y_2		.	.	$Y_2{+}Y_3$
Y_3			.	Y_3
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-e_1$
e_2		.	.	$-e_1{-}e_2$
e_3			.	$-e_2{-}e_3$
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_3, X_2 \to e_2, X_3 \to e_1, X_4 \to -e_4]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^2}$$
$$X_2 = \partial_{x^3}$$
$$X_3 = -\partial_{x^1}$$
$$X_4 = (x^1 - x^3)\,\partial_{x^1} + x^2\,\partial_{x^2} + (x^3 + x^2)\,\partial_{x^3} + \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = \mathrm{e}^{-2\,x^4}\,dx^1\,dx^1 - \frac{1}{2}\,{x^4}^2\mathrm{e}^{-2\,x^4}\,dx^1\,dx^2 + x^4\,\mathrm{e}^{-2\,x^4}\,dx^1\,dx^3 + \frac{1}{4}\,{x^4}^4\mathrm{e}^{-2\,x^4}\,dx^2\,dx^2 - \frac{1}{2}\,{x^4}^3\mathrm{e}^{-2\,x^4}\,dx^2\,dx^3 + {x^4}^2\mathrm{e}^{-2\,x^4}\,dx^3\,dx^3$$

$$\sigma^2 = \frac{1}{2}\,\mathrm{e}^{-2\,x^4}\,dx^1\,dx^2 - \frac{1}{2}\,{x^4}^2\mathrm{e}^{-2\,x^4}\,dx^2\,dx^2 + \frac{1}{2}\,x^4\,\mathrm{e}^{-2\,x^4}\,dx^2\,dx^3$$

$$\sigma^3 = -\frac{1}{2}\,x^4\,\mathrm{e}^{-2\,x^4}\,dx^1\,dx^2 + \frac{1}{2}\,\mathrm{e}^{-2\,x^4}\,dx^1\,dx^3 + \frac{1}{2}\,{x^4}^3\mathrm{e}^{-2\,x^4}\,dx^2\,dx^2 - 3/4\,{x^4}^2\mathrm{e}^{-2\,x^4}\,dx^2\,dx^3 + x^4\,\mathrm{e}^{-2\,x^4}\,dx^3\,dx^3$$

$$\sigma^4 = \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^1\,dx^4 - \frac{1}{4}\,{x^4}^2\mathrm{e}^{-x^4}\,dx^2\,dx^4 + \frac{1}{2}\,x^4\,\mathrm{e}^{-x^4}\,dx^3\,dx^4$$

$$\sigma^5 = \mathrm{e}^{-2\,x^4}\,dx^2\,dx^2$$

$$\sigma^6 = -x^4\,\mathrm{e}^{-2\,x^4}\,dx^2\,dx^2 + \frac{1}{2}\,\mathrm{e}^{-2\,x^4}\,dx^2\,dx^3$$

$$\sigma^7 = {x^4}^2\mathrm{e}^{-2\,x^4}\,dx^2\,dx^2 - x^4\,\mathrm{e}^{-2\,x^4}\,dx^2\,dx^3 + \mathrm{e}^{-2\,x^4}\,dx^3\,dx^3$$

$$\sigma^8 = \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^2\,dx^4$$

$$\sigma^9 = -\frac{1}{2}\,x^4\,\mathrm{e}^{-x^4}\,dx^2\,dx^4 + \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^3\,dx^4$$

$\sigma^{10} = dx^4\, dx^4$

6. Petrov Reference: [[32, 45, 1]]

[4, 4, 15]

1. Reference : s(4,5), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	$b\,Y_1+Y_2$
Y_2		.	.	$b\,Y_2-Y_1$
Y_3			.	$a\,Y_3$
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-a\,e_1$
e_2		.	.	$-b\,e_2+e_3$
e_3			.	$-b\,e_3-e_2$
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_3, X_2 \to e_2, X_3 \to e_1, X_4 \to -e_4]$

4. Vector Fields Γ: $[0 < a]$

$$X_1 = \partial_{x^2}$$
$$X_2 = \partial_{x^3}$$
$$X_3 = -\partial_{x^1}$$
$$X_4 = a\,x^1\,\partial_{x^1} + (b\,x^2 - x^3)\,\partial_{x^2} + (b\,x^3 + x^2)\,\partial_{x^3} + \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = \mathrm{e}^{-2\,a\,x^4}\,dx^1\,dx^1$$
$$\sigma^2 = -\frac{1}{2}\,\sin\left(x^4\right)\mathrm{e}^{-x^4\,(a+b)}\,dx^1\,dx^2 + \frac{1}{2}\,\mathrm{e}^{-x^4\,(a+b)}\cos\left(x^4\right)\,dx^1\,dx^3$$
$$\sigma^3 = \frac{1}{2}\,\mathrm{e}^{-x^4\,(a+b)}\cos\left(x^4\right)\,dx^1\,dx^2 + \frac{1}{2}\,\sin\left(x^4\right)\mathrm{e}^{-x^4\,(a+b)}\,dx^1\,dx^3$$
$$\sigma^4 = \frac{1}{2}\,\mathrm{e}^{-a\,x^4}\,dx^1\,dx^4$$
$$\sigma^5 = \left(\sin\left(x^4\right)\right)^2\mathrm{e}^{-2\,b\,x^4}\,dx^2\,dx^2 - \mathrm{e}^{-2\,b\,x^4}\sin\left(x^4\right)\cos\left(x^4\right)\,dx^2\,dx^3 + \mathrm{e}^{-2\,b\,x^4}\left(\cos\left(x^4\right)\right)^2\,dx^3\,dx^3$$
$$\sigma^6 = -\mathrm{e}^{-2\,b\,x^4}\sin\left(x^4\right)\cos\left(x^4\right)\,dx^2\,dx^2 + \frac{1}{2}\,\cos\left(2\,x^4\right)\mathrm{e}^{-2\,b\,x^4}\,dx^2\,dx^3 + \mathrm{e}^{-2\,b\,x^4}\sin\left(x^4\right)\cos\left(x^4\right)\,dx^3\,dx^3$$
$$\sigma^7 = \mathrm{e}^{-2\,b\,x^4}\left(\cos\left(x^4\right)\right)^2\,dx^2\,dx^2 + \mathrm{e}^{-2\,b\,x^4}\sin\left(x^4\right)\cos\left(x^4\right)\,dx^2\,dx^3 + \left(\sin\left(x^4\right)\right)^2\mathrm{e}^{-2\,b\,x^4}\,dx^3\,dx^3$$
$$\sigma^8 = -\frac{1}{2}\,\sin\left(x^4\right)\mathrm{e}^{-b\,x^4}\,dx^2\,dx^4 + \frac{1}{2}\,\mathrm{e}^{-b\,x^4}\cos\left(x^4\right)\,dx^3\,dx^4$$
$$\sigma^9 = \frac{1}{2}\,\mathrm{e}^{-b\,x^4}\cos\left(x^4\right)\,dx^2\,dx^4 + \frac{1}{2}\,\sin\left(x^4\right)\mathrm{e}^{-b\,x^4}\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[32, 42, 0]]

[4, 4, 16]

1. Reference : s(4,8), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	$(1+a)Y_1$
Y_2		.	Y_1	Y_2
Y_3			.	$a\,Y_3$
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	$(-a-1)e_1$
e_2		.	e_1	$-e_2$
e_3			.	$-a\,e_3$
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to -e_4]$

4. Vector Fields Γ: $[-1 < a, a < 1, a \neq 0]$

$$X_1 = \partial_{x^2}$$
$$X_2 = \partial_{x^3}$$
$$X_3 = -\partial_{x^1} + x^3\,\partial_{x^2}$$
$$X_4 = ((1+a)\,x^1 - x^1)\,\partial_{x^1} + (1+a)\,x^2\,\partial_{x^2} + x^3\,\partial_{x^3} + \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = \mathrm{e}^{-2\,a\,x^4}\,dx^1\,dx^1$$
$$\sigma^2 = \frac{1}{2}\,\mathrm{e}^{-(1+2\,a)x^4}\,dx^1\,dx^2 + \frac{1}{2}\,x^1\,\mathrm{e}^{-(1+2\,a)x^4}\,dx^1\,dx^3$$
$$\sigma^3 = \frac{1}{2}\,\mathrm{e}^{-x^4\,(1+a)}\,dx^1\,dx^3$$
$$\sigma^4 = \frac{1}{2}\,\mathrm{e}^{-a\,x^4}\,dx^1\,dx^4$$
$$\sigma^5 = \mathrm{e}^{-2\,x^4\,(1+a)}\,dx^2\,dx^2 + x^1\,\mathrm{e}^{-2\,x^4\,(1+a)}\,dx^2\,dx^3 + {x^1}^2\mathrm{e}^{-2\,x^4\,(1+a)}\,dx^3\,dx^3$$
$$\sigma^6 = \frac{1}{2}\,\mathrm{e}^{-(2+a)x^4}\,dx^2\,dx^3 + x^1\,\mathrm{e}^{-(2+a)x^4}\,dx^3\,dx^3$$
$$\sigma^7 = \frac{1}{2}\,\mathrm{e}^{-x^4\,(1+a)}\,dx^2\,dx^4 + \frac{1}{2}\,x^1\,\mathrm{e}^{-x^4\,(1+a)}\,dx^3\,dx^4$$
$$\sigma^8 = \mathrm{e}^{-2\,x^4}\,dx^3\,dx^3$$
$$\sigma^9 = \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[32, 34, 0]]

[4, 4, 17]

1. Reference : s(4,8) a=1, Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	$-2\,Y_1$
Y_2		.	Y_1	$-Y_2$
Y_3			.	$-Y_3$
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-2\,e_1$
e_2		.	e_1	$-e_2$
e_3			.	$-e_3$
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^1}$$
$$X_2 = -\frac{1}{2}\,x^3\,\partial_{x^1} + \partial_{x^2}$$
$$X_3 = \frac{1}{2}\,x^2\,\partial_{x^1} + \partial_{x^3}$$
$$X_4 = -2\,x^1\,\partial_{x^1} - x^2\,\partial_{x^2} - x^3\,\partial_{x^3} + \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = 4\,e^{4\,x^4}\,dx^1\,dx^1 - 2\,e^{4\,x^4}\,x^3\,dx^1\,dx^2 + 2\,e^{4\,x^4}\,x^2\,dx^1\,dx^3 + e^{4\,x^4}\,{x^3}^2\,dx^2\,dx^2 - e^{4\,x^4}\,x^2\,x^3\,dx^2\,dx^3 + {x^2}^2 e^{4\,x^4}\,dx^3\,dx^3$$
$$\sigma^2 = e^{3\,x^4}\,dx^1\,dx^2 - e^{3\,x^4}\,x^3\,dx^2\,dx^2 + \frac{1}{2}\,x^2\,e^{3\,x^4}\,dx^2\,dx^3$$
$$\sigma^3 = e^{3\,x^4}\,dx^1\,dx^3 - \frac{1}{2}\,e^{3\,x^4}\,x^3\,dx^2\,dx^3 + x^2\,e^{3\,x^4}\,dx^3\,dx^3$$
$$\sigma^4 = e^{2\,x^4}\,dx^1\,dx^4 - \frac{1}{2}\,e^{2\,x^4}\,x^3\,dx^2\,dx^4 + \frac{1}{2}\,x^2\,e^{2\,x^4}\,dx^3\,dx^4$$
$$\sigma^5 = e^{2\,x^4}\,dx^2\,dx^2$$
$$\sigma^6 = \frac{1}{2}\,e^{2\,x^4}\,dx^2\,dx^3$$
$$\sigma^7 = \frac{1}{2}\,e^{x^4}\,dx^2\,dx^4$$
$$\sigma^8 = e^{2\,x^4}\,dx^3\,dx^3$$
$$\sigma^9 = \frac{1}{2}\,e^{x^4}\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[32, 34, 2]]

[4, 4, 18]

1. Reference : s(4,6), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	.
Y_2		.	Y_1	Y_2
Y_3			.	$-Y_3$
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	.
e_2		.	e_1	$-e_2$
e_3			.	e_3
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to -e_4]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^2}$$
$$X_2 = \partial_{x^3}$$
$$X_3 = -\partial_{x^1} + x^3\,\partial_{x^2}$$
$$X_4 = -x^1\,\partial_{x^1} + x^3\,\partial_{x^3} + \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = \mathrm{e}^{2\,x^4}\,dx^1\,dx^1$$
$$\sigma^2 = \frac{1}{2}\,\mathrm{e}^{x^4}\,dx^1\,dx^2 + \frac{1}{2}\,\mathrm{e}^{x^4}\,x^1\,dx^1\,dx^3$$
$$\sigma^3 = \frac{1}{2}\,dx^1\,dx^3$$
$$\sigma^4 = \frac{1}{2}\,\mathrm{e}^{x^4}\,dx^1\,dx^4$$
$$\sigma^5 = dx^2\,dx^2 + x^1\,dx^2\,dx^3 + {x^1}^2\,dx^3\,dx^3$$
$$\sigma^6 = \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^2\,dx^3 + \mathrm{e}^{-x^4}\,x^1\,dx^3\,dx^3$$
$$\sigma^7 = \frac{1}{2}\,dx^2\,dx^4 + \frac{1}{2}x^1\,dx^3\,dx^4$$
$$\sigma^8 = \mathrm{e}^{-2\,x^4}\,dx^3\,dx^3$$
$$\sigma^9 = \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[32, 34, 1]]

[4, 4, 19]

1. Reference : s(4,11), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	Y_1
Y_2		.	Y_1	Y_2
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-e_1$
e_2		.	e_1	$-e_2$
e_3			.	.
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to -e_4]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^2}$$
$$X_2 = \partial_{x^3}$$
$$X_3 = -\partial_{x^1} + x^3\,\partial_{x^2}$$
$$X_4 = \alpha\,\partial_{x^1} + x^2\,\partial_{x^2} + x^3\,\partial_{x^3} + \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1$$
$$\sigma^2 = \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^1\,dx^2 - \frac{1}{2}\,\mathrm{e}^{-x^4}\left(\alpha\,x^4 - x^1\right)\,dx^1\,dx^3$$
$$\sigma^3 = \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^1\,dx^3$$
$$\sigma^4 = \frac{1}{2}\,dx^1\,dx^4$$
$$\sigma^5 = \frac{1}{2}\,\mathrm{e}^{-2\,x^4}\,dx^2\,dx^2 - \frac{1}{2}\,\mathrm{e}^{-2\,x^4}\left(\alpha\,x^4 - x^1\right)\,dx^2\,dx^3 + \frac{1}{2}\,\mathrm{e}^{-2\,x^4}\left(\alpha^2{x^4}^2 - 2\,\alpha\,x^1\,x^4 + {x^1}^2\right)\,dx^3\,dx^3$$
$$\sigma^6 = \frac{1}{2}\,\mathrm{e}^{-2\,x^4}\,dx^2\,dx^3 - \mathrm{e}^{-2\,x^4}\left(\alpha\,x^4 - x^1\right)\,dx^3\,dx^3$$
$$\sigma^7 = \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^2\,dx^4 - \frac{1}{2}\,\mathrm{e}^{-x^4}\left(\alpha\,x^4 - x^1\right)\,dx^3\,dx^4$$
$$\sigma^8 = \mathrm{e}^{-2\,x^4}\,dx^3\,dx^3$$
$$\sigma^9 = \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[32, 35, 0]]

[4, 4, 20]

1. Reference : s(4,10), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	$2\,Y_1$
Y_2		.	Y_1	Y_2
Y_3			.	Y_2+Y_3
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-2\,e_1$
e_2		.	e_1	$-e_2$
e_3			.	$-e_2-e_3$
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to -e_4]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^2}$$
$$X_2 = \partial_{x^3}$$
$$X_3 = -\partial_{x^1} + x^3\,\partial_{x^2}$$
$$X_4 = x^1\,\partial_{x^1} + (2\,x^2 + \frac{1}{2}\,\left(x^1\right)^2)\,\partial_{x^2} + (x^3 - x^1)\,\partial_{x^3} + \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = x^4\,\mathrm{e}^{-2\,x^4}\,dx^1\,dx^1 + \frac{1}{2}\,\mathrm{e}^{-2\,x^4}\,dx^1\,dx^3$$
$$\sigma^2 = x^4\,\left(x^4-2\right)\mathrm{e}^{-2\,x^4}\,dx^1\,dx^1 + \mathrm{e}^{-2\,x^4}\,\left(-1+x^4\right)\,dx^1\,dx^3 + \mathrm{e}^{-2\,x^4}\,dx^3\,dx^3$$
$$\sigma^3 = \mathrm{e}^{-2\,x^4}\,dx^1\,dx^1$$
$$\sigma^4 = \frac{1}{2}\,\mathrm{e}^{-3\,x^4}\,dx^1\,dx^2 + \frac{1}{2}\,\mathrm{e}^{-3\,x^4}\,x^1\,dx^1\,dx^3$$
$$\sigma^5 = \frac{1}{2}\,\left(x^4+1\right)\mathrm{e}^{-3\,x^4}\,dx^1\,dx^2 + \frac{1}{2}\,x^1\,\left(x^4+1\right)\mathrm{e}^{-3\,x^4}\,dx^1\,dx^3 + \frac{1}{2}\,\mathrm{e}^{-3\,x^4}\,dx^2\,dx^3 + \mathrm{e}^{-3\,x^4}\,x^1\,dx^3\,dx^3$$
$$\sigma^6 = \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^1\,dx^4$$
$$\sigma^7 = \frac{1}{2}\,x^4\,\mathrm{e}^{-x^4}\,dx^1\,dx^4 + \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^3\,dx^4$$
$$\sigma^8 = \mathrm{e}^{-4\,x^4}\,dx^2\,dx^2 + \mathrm{e}^{-4\,x^4}\,x^1\,dx^2\,dx^3 + {x^1}^2\,\mathrm{e}^{-4\,x^4}\,dx^3\,dx^3$$
$$\sigma^9 = \frac{1}{2}\,\mathrm{e}^{-2\,x^4}\,dx^2\,dx^4 + \frac{1}{2}\,x^1\,\mathrm{e}^{-2\,x^4}\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[32, 36, 0]]

[4, 4, 21]

1. Reference : s(4,9), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	$-2\,a\,Y_1$
Y_2		.	Y_1	$-a\,Y_2+Y_3$
Y_3			.	$-a\,Y_3-Y_2$
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	$-2\,a\,e_1$
e_2		.	e_1	$-a\,e_2+e_3$
e_3			.	$-a\,e_3-e_2$
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4]$

4. Vector Fields Γ: $[a \neq 0]$

$$X_1 = \partial_{x^1}$$
$$X_2 = -\frac{1}{2}\,\frac{a\,x^3 - x^2}{a}\,\partial_{x^1} + \partial_{x^2}$$
$$X_3 = \frac{1}{2}\,\frac{a\,x^2 + x^3}{a}\,\partial_{x^1} + \partial_{x^3}$$
$$X_4 = -2\,a\,x^1\,\partial_{x^1} + (-a\,x^2 - x^3)\,\partial_{x^2} + (-a\,x^3 + x^2)\,\partial_{x^3} + \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\begin{aligned}
\sigma^1 = {} & 2\,\mathrm{e}^{4\,a\,x^4}\,a^2\,dx^1\,dx^1 - a\,\mathrm{e}^{4\,a\,x^4}\left(a\,x^3 + x^2\right)\,dx^1\,dx^2 + \mathrm{e}^{4\,a\,x^4}\left(a\,x^2 - x^3\right)\,a\,dx^1\,dx^3 \\
& + \frac{1}{2}\,\mathrm{e}^{4\,a\,x^4}\left(a^2{x^3}^2 + 2\,a\,x^2\,x^3 + {x^2}^2\right)\,dx^2\,dx^2 \\
& - \frac{1}{2}\,\mathrm{e}^{4\,a\,x^4}\left(a^2x^2\,x^3 + a\,{x^2}^2 - a\,{x^3}^2 - x^2\,x^3\right)\,dx^2\,dx^3 \\
& + \frac{1}{2}\,\mathrm{e}^{4\,a\,x^4}\left(a^2{x^2}^2 - 2\,a\,x^2\,x^3 + {x^3}^2\right)\,dx^3\,dx^3
\end{aligned}$$

$$\begin{aligned}
\sigma^2 = {} & \mathrm{e}^{3\,a\,x^4}\sin\left(x^4\right)\,a\,dx^1\,dx^2 - a\,\mathrm{e}^{3\,a\,x^4}\cos\left(x^4\right)\,dx^1\,dx^3 - \mathrm{e}^{3\,a\,x^4}\sin\left(x^4\right)\left(a\,x^3 + x^2\right)\,dx^2\,dx^2 \\
& + \frac{1}{2}\,\mathrm{e}^{3\,a\,x^4}\left(\sin\left(x^4\right)\,a\,x^2 + \cos\left(x^4\right)\,a\,x^3 - \sin\left(x^4\right)\,x^3 + \cos\left(x^4\right)\,x^2\right)\,dx^2\,dx^3 \\
& - \mathrm{e}^{3\,a\,x^4}\cos\left(x^4\right)\left(a\,x^2 - x^3\right)\,dx^3\,dx^3
\end{aligned}$$

$$\begin{aligned}
\sigma^3 = {} & -a\,\mathrm{e}^{3\,a\,x^4}\cos\left(x^4\right)\,dx^1\,dx^2 - \mathrm{e}^{3\,a\,x^4}\sin\left(x^4\right)\,a\,dx^1\,dx^3 + \mathrm{e}^{3\,a\,x^4}\cos\left(x^4\right)\left(a\,x^3 + x^2\right)\,dx^2\,dx^2 \\
& + \frac{1}{2}\,\mathrm{e}^{3\,a\,x^4}\left(a\,\sin\left(x^4\right)\,x^3 - \cos\left(x^4\right)\,a\,x^2 + \sin\left(x^4\right)\,x^2 + \cos\left(x^4\right)\,x^3\right)\,dx^2\,dx^3 \\
& - \mathrm{e}^{3\,a\,x^4}\sin\left(x^4\right)\left(a\,x^2 - x^3\right)\,dx^3\,dx^3
\end{aligned}$$

$$\sigma^4 = -a\,\mathrm{e}^{2\,a\,x^4}\,dx^1\,dx^4 + \frac{1}{2}\,\mathrm{e}^{2\,a\,x^4}\left(a\,x^3 + x^2\right)\,dx^2\,dx^4 - \frac{1}{2}\,\mathrm{e}^{2\,a\,x^4}\left(a\,x^2 - x^3\right)\,dx^3\,dx^4$$

$$\sigma^5 = -\mathrm{e}^{2\,a\,x^4}\sin\left(2\,x^4\right)\,dx^2\,dx^2 + \mathrm{e}^{2\,a\,x^4}\cos\left(2\,x^4\right)\,dx^2\,dx^3 + \mathrm{e}^{2\,a\,x^4}\sin\left(2\,x^4\right)\,dx^3\,dx^3$$

$$\sigma^6 = \mathrm{e}^{2\,a\,x^4}\,dx^2\,dx^2 + \mathrm{e}^{2\,a\,x^4}\,dx^3\,dx^3$$

$$\sigma^7 = -\mathrm{e}^{2\,a\,x^4}\cos\left(2\,x^4\right)\,dx^2\,dx^2 - \mathrm{e}^{2\,a\,x^4}\sin\left(2\,x^4\right)\,dx^2\,dx^3 + \mathrm{e}^{2\,a\,x^4}\cos\left(2\,x^4\right)\,dx^3\,dx^3$$

$$\sigma^8 = -\frac{1}{2}\,\mathrm{e}^{a\,x^4}\sin\left(x^4\right)\,dx^2\,dx^4 + \frac{1}{2}\,\mathrm{e}^{a\,x^4}\cos\left(x^4\right)\,dx^3\,dx^4$$

$$\sigma^9 = \frac{1}{2}\,\mathrm{e}^{a\,x^4}\cos\left(x^4\right)\,dx^2\,dx^4 + \frac{1}{2}\,\mathrm{e}^{a\,x^4}\sin\left(x^4\right)\,dx^3\,dx^4$$

$\sigma^{10} = dx^4\, dx^4$

6. Petrov Reference: [[32, 37, 0]]

[4, 4, 22]

1. Reference : s(4,7), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	.	.
Y_2		.	Y_1	Y_3
Y_3			.	$-Y_2$
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	.	.
e_2		.	e_1	e_3
e_3			.	$-e_2$
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^2}$$
$$X_2 = \partial_{x^3}$$
$$X_3 = -\partial_{x^1} + x^3\,\partial_{x^2}$$
$$X_4 = -x^3\,\partial_{x^1} + (\frac{1}{2}\,x^6 - \frac{1}{2}\,\left(x^1\right)^2)\,\partial_{x^2} + x^1\,\partial_{x^3} + \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = -\sin\left(2\,x^4\right)\,dx^1\,dx^1 + (2\,\left(\cos\left(x^4\right)\right)^2 - 1)\,dx^1\,dx^3 + 2\,\sin\left(x^4\right)\cos\left(x^4\right)\,dx^3\,dx^3$$
$$\sigma^2 = dx^1\,dx^1 + dx^3\,dx^3$$
$$\sigma^3 = -\cos\left(2\,x^4\right)\,dx^1\,dx^1 - 2\,\sin\left(x^4\right)\cos\left(x^4\right)\,dx^1\,dx^3 + (2\,\left(\cos\left(x^4\right)\right)^2 - 1)\,dx^3\,dx^3$$
$$\sigma^4 = -\frac{1}{2}\,\sin\left(x^4\right)\,dx^1\,dx^2 - \frac{1}{2}\,\sin\left(x^4\right)x^1\,dx^1\,dx^3 + \frac{1}{2}\,\cos\left(x^4\right)\,dx^2\,dx^3 + \cos\left(x^4\right)x^1\,dx^3\,dx^3$$
$$\sigma^5 = \frac{1}{2}\,\cos\left(x^4\right)\,dx^1\,dx^2 + \frac{1}{2}\,\cos\left(x^4\right)x^1\,dx^1\,dx^3 + \frac{1}{2}\,\sin\left(x^4\right)\,dx^2\,dx^3 + \sin\left(x^4\right)x^1\,dx^3\,dx^3$$
$$\sigma^6 = -\frac{1}{2}\,\sin\left(x^4\right)\,dx^1\,dx^4 + \frac{1}{2}\,\cos\left(x^4\right)\,dx^3\,dx^4$$
$$\sigma^7 = \frac{1}{2}\,\cos\left(x^4\right)\,dx^1\,dx^4 + \frac{1}{2}\,\sin\left(x^4\right)\,dx^3\,dx^4$$
$$\sigma^8 = \frac{1}{2}\,dx^2\,dx^2 + \frac{1}{2}x^1\,dx^2\,dx^3 + \frac{1}{2}\,x^{1^2}\,dx^3\,dx^3$$
$$\sigma^9 = \frac{1}{2}\,dx^2\,dx^4 + \frac{1}{2}x^1\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[32, 37, 1]]

[4, 4, 23]

1. Reference : s(4,12), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4
Y_1	.	.	Y_1	Y_2
Y_2		.	Y_2	$-Y_1$
Y_3			.	.
Y_4				.

	e_1	e_2	e_3	e_4
e_1	.	.	$-e_1$	e_2
e_2		.	$-e_2$	$-e_1$
e_3			.	.
e_4				.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to -e_3, X_4 \to e_4]$

4. Vector Fields Γ:

$$X_1 = \partial_{x^2}$$
$$X_2 = \partial_{x^3}$$
$$X_3 = -\partial_{x^1} + x^2\,\partial_{x^2} + x^3\,\partial_{x^3}$$
$$X_4 = \alpha\,\partial_{x^1} - x^3\,\partial_{x^2} + x^2\,\partial_{x^3} + \partial_{x^4}$$

5. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1$$
$$\sigma^2 = -\frac{1}{2}\sin\left(x^4\right)\mathrm{e}^{-\alpha x^4+x^1}\,dx^1\,dx^2 + \frac{1}{2}\cos\left(x^4\right)\mathrm{e}^{-\alpha x^4+x^1}\,dx^1\,dx^3$$
$$\sigma^3 = \frac{1}{2}\cos\left(x^4\right)\mathrm{e}^{-\alpha x^4+x^1}\,dx^1\,dx^2 + \frac{1}{2}\sin\left(x^4\right)\mathrm{e}^{-\alpha x^4+x^1}\,dx^1\,dx^3$$
$$\sigma^4 = \frac{1}{2}\,dx^1\,dx^4$$
$$\sigma^5 = -\cos\left(2\,x^4\right)\mathrm{e}^{-2\,\alpha x^4+2\,x^1}\,dx^2\,dx^2 - \sin\left(2\,x^4\right)\mathrm{e}^{-2\,\alpha x^4+2\,x^1}\,dx^2\,dx^3 + \cos\left(2\,x^4\right)\mathrm{e}^{-2\,\alpha x^4+2\,x^1}\,dx^3\,dx^3$$
$$\sigma^6 = -\sin\left(2\,x^4\right)\mathrm{e}^{-2\,\alpha x^4+2\,x^1}\,dx^2\,dx^2 + \cos\left(2\,x^4\right)\mathrm{e}^{-2\,\alpha x^4+2\,x^1}\,dx^2\,dx^3 + \sin\left(2\,x^4\right)\mathrm{e}^{-2\,\alpha x^4+2\,x^1}\,dx^3\,dx^3$$
$$\sigma^7 = \mathrm{e}^{-2\,\alpha x^4+2\,x^1}\,dx^2\,dx^2 + \mathrm{e}^{-2\,\alpha x^4+2\,x^1}\,dx^3\,dx^3$$
$$\sigma^8 = -\frac{1}{2}\sin\left(x^4\right)\mathrm{e}^{-\alpha x^4+x^1}\,dx^2\,dx^4 + \frac{1}{2}\cos\left(x^4\right)\mathrm{e}^{-\alpha x^4+x^1}\,dx^3\,dx^4$$
$$\sigma^9 = \frac{1}{2}\cos\left(x^4\right)\mathrm{e}^{-\alpha x^4+x^1}\,dx^2\,dx^4 + \frac{1}{2}\sin\left(x^4\right)\mathrm{e}^{-\alpha x^4+x^1}\,dx^3\,dx^4$$
$$\sigma^{10} = dx^4\,dx^4$$

6. Petrov Reference: [[32, 40, 0]]

A.2.5 G_5 on V_4

A.2.5.1 Non-reductive

[5, 4, -6]

1. Reference : A3 epsilon=-1, Fels

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	.	$-Y_5$	.	.	$-Y_2$
Y_2		.	.	Y_2	Y_3
Y_3			.	$2\,Y_3$	.
Y_4				.	$-Y_5$
Y_5					.

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	$2\,e_1$	.
e_2		.	e_1	e_2	e_3
e_3			.	e_3	e_2
e_4				.	.
e_5					.

3. Isomorphisms:

$[X_1 \to Y_3, X_2 \to Y_2, X_3 \to Y_5, X_4 \to Y_4, X_5 \to Y_1]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4, X_5 \to e_5]$

4. Isotropy: F14 $[\,e_3]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^4} \qquad X_4 = \partial_{x^1} + x^3\,\partial_{x^3} + 2\,x^4\,\partial_{x^4}$$
$$X_2 = (\frac{1}{2} + \frac{1}{2}\,\mathrm{e}^{2\,x^2})\,\partial_{x^3} - x^3\,\partial_{x^4} \qquad X_5 = \partial_{x^2} + x^3\,\partial_{x^3}$$
$$X_3 = (\frac{1}{2} - \frac{1}{2}\,\mathrm{e}^{2\,x^2})\,\partial_{x^3} + x^3\,\partial_{x^4}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1$$
$$\sigma^2 = \frac{1}{2}\,dx^1\,dx^2$$
$$\sigma^3 = dx^2\,dx^2$$
$$\sigma^4 = \mathrm{e}^{-2\,x^1}\,dx^2\,dx^4 + \mathrm{e}^{-2\,x^2-2\,x^1}\,dx^3\,dx^3$$

8. Determinants :

$\det(g) = -s_1 {s_4}^3 \mathrm{e}^{-6\,x^1-2\,x^2}$

9. Normalizers:

$$\Phi_1 = [x^1 = \xi_2 + x^1, x^2 = x^2, x^3 = x^3, x^4 = \mathrm{e}^{2\,\xi_2+2\,x^1}\xi_1 + x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1, 0, s_3, e], [e^2 = 1]]$

11. Petrov Reference: [[33, 5, 0], [33, 5, 1], [33, 5, 2], [33, 8, 0], [33, 8, 1], [33, 8, 2]]

[5, 4, -5]

1. Reference : A3 epsilon=1, Fels

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	.	Y_5	.	.	$-Y_2$
Y_2		.	.	Y_2	Y_3
Y_3			.	$2\,Y_3$	.
Y_4				.	$-Y_5$
Y_5					.

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	$2\,e_1$	.
e_2		.	e_1	e_2	$-e_3$
e_3			.	e_3	e_2
e_4				.	.
e_5					.

3. Isomorphisms:

$[X_1 \to Y_3, X_2 \to Y_2, X_3 \to Y_5, X_4 \to Y_4, X_5 \to Y_1]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4, X_5 \to e_5]$

4. Isotropy: F14 $[e_3]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^4}$$
$$X_2 = -\partial_{x^3} + x^5\,\partial_{x^4}$$
$$X_3 = \tan\left(x^2\right)\,\partial_{x^3} - x^3\left(\tan\left(x^2\right)x^2 + 1\right)\,\partial_{x^4}$$
$$X_4 = \partial_{x^1} + x^3\,\partial_{x^3} + 2\,x^4\,\partial_{x^4}$$
$$X_5 = \partial_{x^2} + x^3\tan\left(x^2\right)\,\partial_{x^3} - x^6\left(\tan\left(x^2\right)x^2 + 1\right)\,\partial_{x^4}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1$$
$$\sigma^2 = \frac{1}{2}\,dx^1\,dx^2$$
$$\sigma^3 = dx^2\,dx^2$$
$$\sigma^4 = {x^3}^2\mathrm{e}^{-2\,x^1}\,dx^2\,dx^2 + \mathrm{e}^{-2\,x^1}x^2\,x^3\,dx^2\,dx^3 + \mathrm{e}^{-2\,x^1}\,dx^2\,dx^4 + \left(\cos\left(x^2\right)\right)^2\mathrm{e}^{-2\,x^1}\,dx^3\,dx^3$$

8. Determinants :

$\det(g) = -s_1{s_4}^3\mathrm{e}^{-6\,x^1}\left(\cos\left(x^2\right)\right)^2$

9. Normalizers:

$$\Phi_1 = [x^1 = \xi_2 + x^1, x^2 = x^2, x^3 = x^3, x^4 = \mathrm{e}^{2\,\xi_2}\mathrm{e}^{2\,x^1}\xi_1 + x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1, 0, s_3, e], [e^2 = 1]]$

11. Petrov Reference: [[33, 7, 0], [33, 7, 1], [33, 7, 2]]

[5, 4, -4]

1. Reference : A2 alpha = 2, Fels

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	.	.	.	Y_1	$-Y_2$
Y_2		.	.	$2\,Y_2$	Y_3
Y_3			.	$3\,Y_3$	.
Y_4				.	$-Y_5$
Y_5					.

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$3\,e_1$
e_2		.	.	e_1	$2\,e_2$
e_3			.	e_2	e_3
e_4				.	e_4
e_5					.

3. Isomorphisms:

$[X_1 \to Y_3, X_2 \to Y_2, X_3 \to -Y_1, X_4 \to Y_5, X_5 \to Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4, X_5 \to e_5]$

4. Isotropy: F14 $[\,e_4]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2} \qquad X_4 = x^3\,\partial_{x^2} - x^1\,\partial_{x^3}$$
$$X_2 = \partial_{x^3} \qquad X_5 = x^1\,\partial_{x^1} + 3\,x^2\,\partial_{x^2} + 2\,x^3\,\partial_{x^3} + \partial_{x^4}$$
$$X_3 = -\,\partial_{x^1}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = \mathrm{e}^{-2\,x^4}\,dx^1\,dx^1$$
$$\sigma^2 = \mathrm{e}^{-4\,x^4}\,dx^1\,dx^2 + \mathrm{e}^{-4\,x^4}\,dx^3\,dx^3$$
$$\sigma^3 = \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^1\,dx^4$$
$$\sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -{s_2}^3 \mathrm{e}^{-12\,x^4} s_4$

9. Normalizers:

$$\Phi_1 = [x^1 = \frac{x^1}{\xi_1}, x^2 = \left(\mathrm{e}^{3\,x^4}\xi_3 + x^2\right)\xi_1, x^3 = x^3, x^4 = x^4 + \xi_2]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[e_1, e_2, 0, s_4], [{e_1}^2 = 1, {e_2}^2 = 1]]$

11. Petrov Reference: [[33, 30, 0]]

[5, 4, -3]

1. Reference : A2 alpha =1, Fels

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	.	.	.	.	$-Y_2$
Y_2		.	.	Y_2	Y_3
Y_3			.	$2\,Y_3$	.
Y_4				.	$-Y_5$
Y_5					.

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$2\,e_1$
e_2		.	.	e_1	e_2
e_3			.	e_2	.
e_4				.	e_4
e_5					.

3. Isomorphisms:

$[X_1 \to Y_3, X_2 \to Y_2, X_3 \to -Y_1, X_4 \to Y_5, X_5 \to Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4, X_5 \to e_5]$

4. Isotropy: F14 $[\,e_4]$

5. Vector Fields Γ:

$$X_1 = x^1\,\partial_{x^4} \qquad X_4 = (x^1-1)\,\partial_{x^2} + x^2\,\partial_{x^4}$$
$$X_2 = \partial_{x^2} - x^2\,\partial_{x^4} \qquad X_5 = x^2\,\partial_{x^2} + \partial_{x^3} + 2\,x^4\,\partial_{x^4}$$
$$X_3 = \partial_{x^1} + 3/2\,\frac{x^4}{x^1}\,\partial_{x^4}$$

6. Base Point: $[1, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1$$
$$\sigma^2 = \frac{\mathrm{e}^{-2\,x^3}\left({x^2}^2+2\,x^4\right)}{{x^1}^2}\,dx^1\,dx^1 - \frac{\mathrm{e}^{-2\,x^3}x^2}{x^1}\,dx^1\,dx^2 - \frac{\mathrm{e}^{-2\,x^3}}{x^1}\,dx^1\,dx^4 + \mathrm{e}^{-2\,x^3}\,dx^2\,dx^2$$
$$\sigma^3 = \frac{1}{2}\,dx^1\,dx^3$$
$$\sigma^4 = dx^3\,dx^3$$

8. Determinants :

$\det(g) = -\frac{{s_2}^3\mathrm{e}^{-6\,x^3}s_4}{{x^1}^2}$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1\mathrm{e}^{-\xi_1} - \mathrm{e}^{-\xi_1} + 1, x^2 = x^2, x^3 = x^3 + \xi_3,$$
$$x^4 = \frac{1}{2}\,\frac{2\,\mathrm{e}^{2\,x^3+\xi_1+2\,\xi_3}\xi_2 x^1 + 2\,\left(x^1\right)^2\mathrm{e}^{2\,x^3+2\,\xi_3}\xi_2 + 3\,x^4\mathrm{e}^{\xi_1} - 2\,\mathrm{e}^{2\,x^3+2\,\xi_3}x^1\xi_2 + 2\,x^1x^4 - 3\,x^4}{x^1}]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[e_1, e_2, 0, s_4], [{e_1}^2 = 1, {e_2}^2 = 1]]$

11. Petrov Reference: [[33, 6, 0], [33, 6, 1], [33, 6, 2], [33, 29, 0]]

[5, 4, -2]

1. Reference : A2, Fels

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	.	.	.	Y_1	$-Y_2$
Y_2		.	.	$\frac{\alpha Y_2}{\alpha-1}$	Y_3
Y_3			.	$\frac{(\alpha+1)Y_3}{\alpha-1}$	.
Y_4				.	$-\frac{Y_5}{\alpha-1}$
Y_5					.

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$(\alpha+1)e_1$
e_2		.	.	e_1	$\alpha\, e_2$
e_3			.	e_2	$(\alpha-1)e_3$
e_4				.	e_4
e_5					.

3. Isomorphisms:

$[X_1 \to -Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to Y_5, X_5 \to Y_4]$
$[X_1 \to e_3, X_2 \to e_2, X_3 \to e_1, X_4 \to e_4, X_5 \to (\alpha-1)^{-1}\, e_5]$

4. Isotropy: F14 $[e_4]$

5. Vector Fields Γ: $[\alpha \neq 0, \alpha \neq -1, \alpha \neq 1, \alpha \neq 2]$

$$X_1 = \partial_{x^2} \qquad X_4 = -x^3\,\partial_{x^1} + x^2\,\partial_{x^3}$$
$$X_2 = \partial_{x^3} \qquad X_5 = \frac{x^1\,(\alpha+1)}{\alpha-1}\,\partial_{x^1} + x^2\,\partial_{x^2}\frac{1}{2}\,x^3\left(\frac{\alpha+1}{\alpha-1}+1\right)\partial_{x^3} + \partial_{x^4}$$
$$X_3 = -\,\partial_{x^1}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = \mathrm{e}^{-2\,\frac{\alpha\,x^4}{\alpha-1}}\,dx^1\,dx^2 + \mathrm{e}^{-2\,\frac{\alpha\,x^4}{\alpha-1}}\,dx^3\,dx^3$$
$$\sigma^2 = \mathrm{e}^{-2\,x^4}\,dx^2\,dx^2$$
$$\sigma^3 = \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^2\,dx^4$$
$$\sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -{s_1}^3\mathrm{e}^{-6\,\frac{\alpha\,x^4}{\alpha-1}}\,s_4$

9. Normalizers:

$$\Phi_1 = [x^1 = \frac{1}{\xi_1}\left(\mathrm{e}^{\frac{(\alpha+1)x^4}{\alpha-1}}\,\xi_3 + x^1\right), x^2 = x^2\xi_1, x^3 = x^3, x^4 = x^4 + \xi_2]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[e_1, e_2, 0, s_4], [{e_1}^2 = 1, {e_2}^2 = 1]]$

11. Petrov Reference: [[33, 12, 0], [33, 13, 0], [33, 29, 1]]

[5, 4, -1]

1. Reference : A1, Fels

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	.	$2Y_1+2Y_3$	$-2Y_2$	$-Y_1-Y_3$	$-Y_2$
Y_2		.	$2Y_3$	.	Y_3
Y_3			.	Y_3	.
Y_4				.	$-Y_5$
Y_5					.

	e_1	e_2	e_3	e_4	e_5
e_1	.	$2e_2$	$-2e_3$	.	.
e_2		.	e_1	.	.
e_3			.	.	.
e_4				.	e_4
e_5					.

3. Isomorphisms:

$[X_1 \to -Y_2 + Y_3, X_2 \to Y_1 + Y_2, X_3 \to \frac{1}{2}Y_3, X_4 \to -\frac{1}{2}Y_3 + Y_5, X_5 \to \frac{1}{2}Y_2 + Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4, X_5 \to e_5]$

4. Isotropy: F14 $[e_3 + e_4]$

5. Vector Fields Γ:

$$X_1 = 2\,\partial_{x^3} \qquad X_4 = (x^4+1)\,\partial_{x^1}$$
$$X_2 = e^{x^3}\,\partial_{x^2} \qquad X_5 = \partial_{x^1} + (-x^4-1)\,\partial_{x^4}$$
$$X_3 = -e^{-x^3}\,\partial_{x^1} + e^{-x^3}x^4\,\partial_{x^2} + 2\,e^{-x^3}x^2\,\partial_{x^3}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = \frac{1}{2}\,dx^1\,dx^2 - \frac{1}{2}x^2\,dx^1\,dx^3 - \frac{1}{2}\,\frac{x^1 + \ln\left(x^4+1\right)}{x^4+1}\,dx^2\,dx^4 - \frac{1}{4}\,dx^3\,dx^3 + \frac{x^2\,\ln\left(x^4+1\right) + x^2\,x^1 - \frac{1}{2}}{2\,x^4+2}\,dx^3\,dx^4$$
$$\sigma^2 = dx^2\,dx^2 - x^2\,dx^2\,dx^3 + {x^2}^2\,dx^3\,dx^3$$
$$\sigma^3 = -\frac{1}{2}\,\left(x^4+1\right)^{-1}\,dx^2\,dx^4 + \frac{1}{2}\,\frac{x^2}{x^4+1}\,dx^3\,dx^4$$
$$\sigma^4 = \left(x^4+1\right)^{-2}\,dx^4\,dx^4$$

8. Determinants :

$\det(g) = \frac{{s_1}^3(s_1+4\,s_4)}{64\,(x^4+1)^2}$

9. Normalizers:

$$\Phi_1 = [x^1 = \ln\left(x^4+1\right)\xi_1 + \xi_1\xi_2 + x^1\xi_1 - \ln\left(x^4+1\right) - 1, x^2 = \frac{x^2}{\xi_1}, x^3 = x^3 - \xi_1, x^4 = x^4\xi_1 + \xi_1 - 1]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1, e, 0, s_4], [e^2 = 1]]$

11. Petrov Reference: missing from Petrov

A.2.5.2 $F12$

[5, 4, 1]

1. Reference : (F12, 4), Rozum

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	.	$-\frac{1}{2}\sqrt{2}Y_3 - Y_5$	.	.	$-Y_2$
Y_2		.	.	.	Y_1
Y_3			.	.	.
Y_4				.	.
Y_5					.

	e_1	e_2	e_3	e_4	e_5
e_1	.	$2\,e_1$	$-2\,e_2$	.	.
e_2		.	$2\,e_3$	.	.
e_3			.	.	.
e_4				.	.
e_5					.

3. Isomorphisms:

$$X_1 \to -Y_2 - \frac{1}{2}\sqrt{2}\,Y_3 - Y_5, \quad X_3 \to \sqrt{2}\,Y_3, \quad X_5 \to -Y_4$$

$$X_2 \to Y_1, \quad X_4 \to Y_2 - \frac{1}{2}\sqrt{2}\,Y_3 - Y_5,$$

$$X_1 \to -\frac{1}{2}\sqrt{2}\,e_1, \quad X_3 \to \sqrt{2}\,e_4, \quad X_5 \to -e_5$$

$$X_2 \to \frac{1}{2}\,e_2, \quad X_4 \to \frac{1}{2}\sqrt{2}\,e_3,$$

4. Isotropy: F12 $[\,e_1 - e_3 - 2\,e_4]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2} \qquad X_4 = \partial_{x^1}$$

$$X_2 = x^2\,\partial_{x^2} + \partial_{x^3} \qquad X_5 = \partial_{x^4}$$

$$X_3 = -\mathrm{e}^{x^3}\,\partial_{x^1} + (-\mathrm{e}^{2\,x^3} + x^4)\,\partial_{x^2} + 2\,x^2\,\partial_{x^3}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = 4\,dx^1\,dx^1 - 2\,\mathrm{e}^{-x^3}\,dx^1\,dx^2 + \mathrm{e}^{-2\,x^3}\,dx^2\,dx^2$$

$$\sigma^2 = -4\,dx^1\,dx^1 + 2\,\mathrm{e}^{-x^3}\,dx^1\,dx^2 + dx^3\,dx^3$$

$$\sigma^3 = -\,dx^1\,dx^4 + \frac{1}{2}\,\mathrm{e}^{-x^3}\,dx^2\,dx^4$$

$$\sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = \mathrm{e}^{-2\,x^3} s_2{}^2\left(4\,s_1 s_4 - 4\,s_2 s_4 - s_3{}^2\right)$

9. Normalizers:

$$\Phi_1 = [x^1 = \xi_2 x^4 + x^1, x^2 = x^2, x^3 = x^3, x^4 = x^4 \xi_1]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1, s_2, 0, e], [e^2 = 1]]$ missing from Petrov

$[[s_1, s_1, -2, 0]]$ Riemann invariants: $r_2 = 0$, $m_4 = 0$

11. Petrov Reference: [[33, 17, 0]]

[5, 4, 2]

1. Reference : (F12, 6), Rozum

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	.	$-Y_3+Y_5$	.	.	$-Y_2$
Y_2		.	.	.	Y_1
Y_3			.	.	.
Y_4				.	.
Y_5					.

	e_1	e_2	e_3	e_4	e_5
e_1	.	e_3	$-e_2$	.	.
e_2		.	e_1	.	.
e_3			.	.	.
e_4				.	.
e_5					.

3. Isomorphisms:

$[X_1 \to Y_2, X_2 \to Y_1, X_3 \to Y_3 - Y_5, X_4 \to Y_3, X_5 \to -Y_4]$
$[X_1 \to e_2, X_2 \to e_3, X_3 \to e_1, X_4 \to e_4, X_5 \to e_4 - e_5]$

4. Isotropy: F12 $[e_1 - e_4]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2}$$
$$X_2 = \cos\left(x^2\right)\,\partial_{x^1} - \frac{\cos\left(x^1\right)\sin\left(x^2\right)}{\sin\left(x^1\right)}\,\partial_{x^2} + \frac{\sin\left(x^2\right)}{\sin\left(x^1\right)}\,\partial_{x^3}$$
$$X_3 = -\sin\left(x^2\right)\,\partial_{x^1} - \frac{\cos\left(x^1\right)\cos\left(x^2\right)}{\sin\left(x^1\right)}\,\partial_{x^2} + \frac{\cos\left(x^2\right)}{\sin\left(x^1\right)}\,\partial_{x^3}$$
$$X_4 = \partial_{x^3}$$
$$X_5 = \partial_{x^4}$$

6. Base Point: $[\pi/2, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1 + (\frac{1}{2} - \frac{1}{2}\cos\left(2\,x^1\right))\,dx^2\,dx^2$$
$$\sigma^2 = dx^1\,dx^1 + dx^2\,dx^2 + \cos\left(x^1\right)\,dx^2\,dx^3 + dx^3\,dx^3$$
$$\sigma^3 = \frac{1}{2}\cos\left(x^1\right)\,dx^2\,dx^4 + \frac{1}{2}\,dx^3\,dx^4$$
$$\sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = \frac{1}{4}\left(\sin\left(x^1\right)\right)^2(s_1+s_2)^2\left(4\,s_2s_4 - {s_3}^2\right)$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1, x^2 = x^2, x^3 = \xi_2 x^4 + x^3, x^4 = x^4\xi_1]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1, s_2, 0, e], [e^2 = 1, s_2 \neq 0]]$	Petrov type D
$[[s_1, 0, 1, 0], [s_2 = 0, s_3 \neq 0]]$	Petrov type N

11. Petrov Reference: [[33, 19, 0], [33, 20, 0]]

[5, 4, 3]

1. Reference : (F12, 8), Rozum

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	.	$-Y_3$	.	.	$-Y_2$
Y_2		.	.	.	Y_1
Y_3			.	.	.
Y_4				.	.
Y_5					.

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	e_1	e_3	.
e_3			.	$-e_2$	.
e_4				.	.
e_5					.

3. Isomorphisms:

$[X_1 \to Y_3, X_2 \to Y_1, X_3 \to -Y_2, X_4 \to Y_5, X_5 \to Y_4]$
$[X_1 \to -e_1, X_2 \to e_2, X_3 \to -e_3, X_4 \to -e_4, X_5 \to e_5]$

4. Isotropy: F12 $[e_4]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2}$$
$$X_2 = \partial_{x^3}$$
$$X_3 = -\partial_{x^1} + x^3\,\partial_{x^2}$$
$$X_4 = -x^3\,\partial_{x^1} + (\frac{1}{2}\,x^6 - \frac{1}{2}\,\left(x^1\right)^2)\,\partial_{x^2} + x^1\,\partial_{x^3}$$
$$X_5 = \partial_{x^4}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = dx^1\,dx^1 + dx^3\,dx^3$$
$$\sigma^2 = dx^2\,dx^2 + x^1\,dx^2\,dx^3 + {x^1}^2\,dx^3\,dx^3$$
$$\sigma^3 = \frac{1}{2}\,dx^2\,dx^4 + \frac{1}{2}x^1\,dx^3\,dx^4$$
$$\sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = \frac{1}{4}\,{s_1}^2\left(4\,s_2 s_4 - {s_3}^2\right)$

9. Normalizers:

$$\Phi_1 = [x^1 = \xi_3 x^1, x^2 = \xi_2 x^4 + x^2{\xi_3}^2, x^3 = x^3\xi_3, x^4 = x^4\xi_1]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$$[[s_1, e_1, 0, e_2], [{e_1}^2 = 1, {e_2}^2 = 1]]$$

11. Petrov Reference: [[33, 23, 0]]

[5, 4, 4]

1. Reference : (F12, 9), Rozum

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	.	Y_4	Y_1	.	$-Y_2$
Y_2		.	Y_2	.	Y_1
Y_3			.	$-2\,Y_4$	.
Y_4				.	.
Y_5					.

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	$-2\,e_1$	.
e_2		.	e_1	$-e_2$	$-e_3$
e_3			.	$-e_3$	e_2
e_4				.	.
e_5					.

3. Isomorphisms:

$[X_1 \to Y_4, X_2 \to -Y_1, X_3 \to -Y_2, X_4 \to -Y_5, X_5 \to Y_3 + Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to -e_5, X_5 \to -e_4]$

4. Isotropy: F12 $[\,e_5]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2}$$
$$X_2 = \partial_{x^3}$$
$$X_3 = -\partial_{x^1} + x^3\,\partial_{x^2}$$
$$X_4 = -x^3\,\partial_{x^1} + (\frac{1}{2}\,x^6 - \frac{1}{2}\,\left(x^1\right)^2)\,\partial_{x^2} + x^1\,\partial_{x^3}$$
$$X_5 = x^1\,\partial_{x^1} + 2\,x^2\,\partial_{x^2} + x^3\,\partial_{x^3} + \partial_{x^4}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = \mathrm{e}^{-2\,x^4}\,dx^1\,dx^1 + \mathrm{e}^{-2\,x^4}\,dx^3\,dx^3$$
$$\sigma^2 = \mathrm{e}^{-4\,x^4}\,dx^2\,dx^2 + \mathrm{e}^{-4\,x^4}\,x^1\,dx^2\,dx^3 + {x^1}^2\mathrm{e}^{-4\,x^4}\,dx^3\,dx^3$$
$$\sigma^3 = \frac{1}{2}\,\mathrm{e}^{-2\,x^4}\,dx^2\,dx^4 + \frac{1}{2}\,x^1\,\mathrm{e}^{-2\,x^4}\,dx^3\,dx^4$$
$$\sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = \frac{1}{4}\,{s_1}^2\mathrm{e}^{-8\,x^4}\left(4\,s_2 s_4 - {s_3}^2\right)$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1, x^2 = \mathrm{e}^{2\,x^4}\xi_2 + x^2, x^3 = x^3, x^4 = x^4 + \xi_1]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1, e, 0, s_4], [e^2 = 1]]$
$[[s_1, 0, e, 0], [e^2 = 1]]$ vanishing Riemann invariants, missing from Petrov

11. Petrov Reference: [[33, 22, 0]]

[5, 4, 5]

1. Reference : (F12, 11), Rozum

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	.	.	.	$-Y_1$	$-Y_2$
Y_2		.	.	$-Y_2$	Y_1
Y_3			.	βY_3	.
Y_4				.	.
Y_5					.

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	βe_1	.
e_2		.	.	$-e_2$	e_3
e_3			.	$-e_3$	$-e_2$
e_4				.	.
e_5					.

3. Isomorphisms:

$[X_1 \to Y_3, X_2 \to Y_1, X_3 \to Y_2, X_4 \to Y_4, X_5 \to -Y_5]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4, X_5 \to e_5]$

4. Isotropy: F12 $[e_5]$

5. Vector Fields Γ: $[\beta \neq 0]$

$$X_1 = \partial_{x^1}$$
$$X_2 = \partial_{x^2}$$
$$X_3 = \partial_{x^3}$$
$$X_4 = \beta x^1 \partial_{x^1} - x^2 \partial_{x^2} - x^3 \partial_{x^3} + \partial_{x^4}$$
$$X_5 = -x^3 \partial_{x^2} + x^2 \partial_{x^3}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = \mathrm{e}^{-2\beta x^4} dx^1 dx^1$$
$$\sigma^2 = \frac{1}{2} \mathrm{e}^{-\beta x^4} dx^1 dx^4$$
$$\sigma^3 = \mathrm{e}^{2x^4} dx^2 dx^2 + \mathrm{e}^{2x^4} dx^3 dx^3$$
$$\sigma^4 = dx^4 dx^4$$

8. Determinants :

$\det(g) = \frac{1}{4} \mathrm{e}^{-2x^4(\beta-2)} s_3{}^2 \left(4 s_1 s_4 - s_2{}^2\right)$

9. Normalizers:

$$\Phi_1 = [x^1 = \left(\mathrm{e}^{\beta x^4} \xi_3 + x^1\right) \xi_1, x^2 = x^2, x^3 = x^3, x^4 = x^4 + \xi_2]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[e_1, 0, e_2, s_4], [e_1{}^2 = 1, e_2{}^2 = 1]]$

11. Petrov Reference: [[33, 31, 0]]

A.2.5.3 $F13$

[5, 4, 6]

1. Reference : (F13, 3), Rozum

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	.	.	.	.	.
Y_2		.	.	.	.
Y_3			.	Y_2	$-Y_4$
Y_4				.	$-Y_3$
Y_5					.

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	.
e_2		.	$-e_1$	$-e_2$	.
e_3			.	e_3	.
e_4				.	.
e_5					.

3. Isomorphisms:

$[X_1 \to -Y_2, X_2 \to -Y_3 + Y_4, X_3 \to \frac{1}{2} Y_3 + \frac{1}{2} Y_4, X_4 \to Y_5, X_5 \to Y_1]$
$[X_1 \to -e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to -e_4, X_5 \to e_5]$

4. Isotropy: F13 $[e_4]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2}$$
$$X_2 = \partial_{x^3}$$
$$X_3 = -\partial_{x^1} + x^3\,\partial_{x^2}$$
$$X_4 = -x^1\,\partial_{x^1} + x^3\,\partial_{x^3}$$
$$X_5 = \partial_{x^4}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = dx^1\,dx^3$$
$$\sigma^2 = dx^2\,dx^2 + x^1\,dx^2\,dx^3 + {x^1}^2\,dx^3\,dx^3$$
$$\sigma^3 = \frac{1}{2}\,dx^2\,dx^4 + \frac{1}{2}x^1\,dx^3\,dx^4$$
$$\sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -\frac{1}{4}\,{s_1}^2\left(4\,s_2 s_4 - {s_3}^2\right)$

9. Normalizers:

$$\Phi_1 = [x^1 = \xi_3 x^1, x^2 = x^4\xi_2 + x^2\xi_3, x^3 = x^3, x^4 = \xi_1 x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[1, s_2, 0, e], [e^2 = 1]]$

11. Petrov Reference: [[33, 21, 1]]

[5, 4, 7]

1. Reference : (F13, 5), Rozum

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	.	.	.	.	.
Y_2		.	.	.	.
Y_3			.	Y_2+Y_5	$-Y_4$
Y_4				.	$-Y_3$
Y_5					.

	e_1	e_2	e_3	e_4	e_5
e_1	.	$2\,e_1$	e_2	.	.
e_2		.	$2\,e_3$	.	.
e_3			.	.	.
e_4				.	.
e_5					.

3. Isomorphisms:

$X_1 \to Y_2 - Y_3 + Y_5,\quad X_3 \to 2\,Y_2,\quad X_5 \to Y_1$
$X_2 \to -Y_4,\quad X_4 \to -Y_2 - Y_3 - Y_5,$

$$X_1 \to \frac{1}{2}\,e_1 + \frac{1}{2}\,e_2 + \frac{1}{2}\,e_3,\quad X_3 \to 2\,e_4,\quad X_5 \to e_5$$
$$X_2 \to -\frac{1}{2}\,e_1 + \frac{1}{2}\,e_3,\quad X_4 \to \frac{1}{2}\,e_1 - \frac{1}{2}\,e_2 + \frac{1}{2}\,e_3,$$

4. Isotropy: F13 $[\,e_2 - 2\,e_4]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^2} \qquad X_4 = \partial_{x^1}$$
$$X_2 = x^2\,\partial_{x^2} + \partial_{x^3} \qquad X_5 = \partial_{x^4}$$
$$X_3 = -\mathrm{e}^{x^3}\,\partial_{x^1} + (\mathrm{e}^{2\,x^3} + x^4)\,\partial_{x^2} + 2\,x^2\,\partial_{x^3}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = 4\,dx^1\,dx^1 + 2\,\mathrm{e}^{-x^3}\,dx^1\,dx^2 + \mathrm{e}^{-2\,x^3}\,dx^2\,dx^2$$
$$\sigma^2 = -4\,dx^1\,dx^1 - 2\,\mathrm{e}^{-x^3}\,dx^1\,dx^2 - dx^3\,dx^3$$
$$\sigma^3 = dx^1\,dx^4 + \frac{1}{2}\,\mathrm{e}^{-x^3}\,dx^2\,dx^4$$
$$\sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -\mathrm{e}^{-2\,x^3}{s_2}^2\left(4\,s_1s_4 - 4\,s_2s_4 - {s_3}^2\right)$

9. Normalizers:

$$\Phi_1 = [x^1 = x^4\xi_2 + x^1, x^2 = x^2, x^3 = x^3, x^4 = \xi_1 x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1, s_2, 0, e], [e^2 = 1]]$

11. Petrov Reference: [[33, 17, 1]]

[5, 4, 8]

1. Reference : (F13, 6), Rozum

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	.	$-Y_1$	.	.	.
Y_2		.	$\frac{1}{2}Y_3-\frac{1}{2}Y_4$	$-\frac{1}{2}Y_3+\frac{1}{2}Y_4$	.
Y_3			.	$2Y_1$	$-Y_4$
Y_4				.	$-Y_3$
Y_5					.

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	$-e_1$	.
e_2		.	e_1	.	e_2
e_3			.	$-e_3$	$-e_3$
e_4				.	.
e_5					.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to \frac{1}{2}Y_3 + \frac{1}{2}Y_4, X_3 \to -\frac{1}{2}Y_3 + \frac{1}{2}Y_4, X_4 \to Y_2, X_5 \to -Y_5]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4, X_5 \to e_5]$

4. Isotropy: F13 $[e_5]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^4} \qquad X_4 = -x^2\,\partial_{x^2} + \partial_{x^3} - x^4\,\partial_{x^4}$$
$$X_2 = \partial_{x^1} - x^2\,\partial_{x^4} \qquad X_5 = x^1\,\partial_{x^1} - x^2\,\partial_{x^2}$$
$$X_3 = \partial_{x^2}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = \frac{1}{2}\,e^{x^3}\,dx^1\,dx^2$$
$$\sigma^2 = {x^1}^2 e^{2\,x^3}\,dx^2\,dx^2 + x^1\,e^{2\,x^3}\,dx^2\,dx^4 + e^{2\,x^3}\,dx^4\,dx^4$$
$$\sigma^3 = \frac{1}{2}\,e^{x^3}x^1\,dx^2\,dx^3 + \frac{1}{2}\,e^{x^3}\,dx^3\,dx^4$$
$$\sigma^4 = dx^3\,dx^3$$

8. Determinants :

$\det(g) = -\frac{1}{16}\,{s_1}^2 e^{4\,x^3}\left(4\,s_2 s_4 - {s_3}^2\right)$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1, x^2 = x^2, x^3 = x^3 + \xi_2, x^4 = e^{-x^3-\xi_2}\xi_1 + x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[e, s_2, 0, s_4], [e^2 = 1]]$

11. Petrov Reference: [[33, 21, 0]]

[5, 4, 9]

1. Reference : (F13, 8), Rozum

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	.	$-a\,Y_1$	.	.	.
Y_2		.	$\frac{1}{2}\,Y_3-\frac{1}{2}\,Y_4$	$-\frac{1}{2}\,Y_3+\frac{1}{2}\,Y_4$	.
Y_3			.	.	$-Y_4$
Y_4				.	$-Y_3$
Y_5					.

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	$-e_1$
e_2		.	.	$-e_2$	.
e_3			.	$-a\,e_3$	$-a\,e_3$
e_4				.	.
e_5					.

3. Isomorphisms:

$[X_1 \to \frac{1}{2}\,Y_3 + \frac{1}{2}\,Y_4, X_2 \to -\frac{1}{2}\,Y_3 + \frac{1}{2}\,Y_4, X_3 \to Y_1, X_4 \to Y_2, X_5 \to Y_2 + Y_5]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4, X_5 \to e_5]$

4. Isotropy: F13 $[e_4 - e_5]$

5. Vector Fields Γ: $[0 < a, a \le 1]$

$$X_1 = \partial_{x^1}$$
$$X_2 = \partial_{x^2}$$
$$X_3 = \partial_{x^3}$$
$$X_4 = -x^2\,\partial_{x^2} - x^3\,a\,\partial_{x^3} + \partial_{x^4}$$
$$X_5 = -x^1\,\partial_{x^1} - x^3\,a\,\partial_{x^3} + \partial_{x^4}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = \frac{1}{2}\,\mathrm{e}^{x^4}\,dx^1\,dx^2$$
$$\sigma^2 = \mathrm{e}^{2\,a\,x^4}\,dx^3\,dx^3$$
$$\sigma^3 = \frac{1}{2}\,\mathrm{e}^{a\,x^4}\,dx^3\,dx^4$$
$$\sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -\frac{1}{16}\,{s_1}^2\mathrm{e}^{2\,x^4\,(1+a)}\left(4\,s_2s_4 - {s_3}^2\right)$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1\mathrm{e}^{\xi_1}, x^2 = x^2, x^3 = \mathrm{e}^{-a\,x^4}\xi_3 + x^3, x^4 = x^4 + \xi_2]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$$[[e_1, e_2, 0, s_4], [{e_1}^2 = 1, {e_2}^2 = 1]]$$

11. Petrov Reference: [[33, 28, 0], [33, 28, 1]]

A.2.5.4 $F14$

[5, 4, 10]

1. Reference : (F14, 1), Rozum

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	.	.	.	$-\epsilon Y_3+Y_1$	$-Y_2$
Y_2		.	.	Y_2	Y_3
Y_3			.	Y_3	.
Y_4				.	.
Y_5					.

	e_1	e_2	e_3	e_4	e_5
e_1	.	.	.	.	e_1
e_2		.	.	e_1	e_2
e_3			.	e_2	ϵe_1+e_3
e_4				.	.
e_5					.

3. Isomorphisms:

$[X_1 \to -Y_3, X_2 \to Y_2, X_3 \to Y_1, X_4 \to -Y_5, X_5 \to Y_4]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4, X_5 \to e_5]$

4. Isotropy: F14 $[e_4]$

5. Vector Fields Γ: $[\epsilon^2 = 1]$

$$X_1 = \partial_{x^1} \qquad X_4 = x^2\,\partial_{x^1} + x^3\,\partial_{x^2}$$
$$X_2 = \partial_{x^2} \qquad X_5 = (x^3\epsilon + x^1)\,\partial_{x^1} + x^2\,\partial_{x^2} + x^3\,\partial_{x^3} + \partial_{x^4}$$
$$X_3 = \epsilon\,\partial_{x^1} + \partial_{x^3}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = -\frac{\epsilon}{2}\,\mathrm{e}^{-2\,x^4}\,dx^1\,dx^3 + \frac{\epsilon}{2}\,\mathrm{e}^{-2\,x^4}\,dx^2\,dx^2 + x^4\,\mathrm{e}^{-2\,x^4}\,dx^3\,dx^3$$
$$\sigma^2 = \mathrm{e}^{-2\,x^4}\,dx^3\,dx^3$$
$$\sigma^3 = \frac{1}{2}\,\mathrm{e}^{-x^4}\,dx^3\,dx^4$$
$$\sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -\frac{1}{8}\,\frac{{s_1}^3\mathrm{e}^{-6\,x^4}s_4}{\epsilon^3}$

9. Normalizers:

$$\Phi_1 = [x^1 = \left(\mathrm{e}^{x^4}\xi_3 + x^1\right)\xi_1, x^2 = x^2\xi_1, x^3 = x^3\xi_1, x^4 = x^4 + \xi_2]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[e, 0, 0, s_4], [e^2 = 1]]$

11. Petrov Reference: [[33, 14, 0], [33, 18, 0], [33, 18, 1]]

[5, 4, 11]

1. Reference : (F14, 2), Rozum

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	.	Y_1	$-Y_2$	.	$-Y_2$
Y_2		.	Y_3	.	Y_3
Y_3			.	.	.
Y_4				.	.
Y_5					.

	e_1	e_2	e_3	e_4	e_5
e_1	.	$2\,e_1$	$-e_2$	.	.
e_2		.	$2\,e_3$	.	.
e_3			.	.	.
e_4				.	.
e_5					.

3. Isomorphisms:

$[X_1 \to -Y_3 + Y_5, X_2 \to 2\,Y_1, X_3 \to -Y_2, X_4 \to Y_4, X_5 \to -Y_3]$
$[X_1 \to e_4, X_2 \to e_1, X_3 \to -\frac{1}{2}\,e_2, X_4 \to e_5, X_5 \to -e_3]$

4. Isotropy: F14 $[\,e_3 + e_4]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^1}$$
$$X_2 = \mathrm{e}^{x^3}\,\partial_{x^2}$$
$$X_3 = \partial_{x^3}$$
$$X_4 = \partial_{x^4}$$
$$X_5 = \mathrm{e}^{-x^3}\,\partial_{x^1} - \mathrm{e}^{-x^3}x^4\,\partial_{x^2} - 2\,\mathrm{e}^{-x^3}x^2\,\partial_{x^3}$$

6. Base Point: $[0,0,0,0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^2 - x^2\,dx^1\,dx^3 - \frac{1}{2}\,dx^3\,dx^3$$
$$\sigma^2 = dx^2\,dx^2 - x^2\,dx^2\,dx^3 + {x^2}^2\,dx^3\,dx^3$$
$$\sigma^3 = -\frac{1}{2}\,dx^2\,dx^4 + \frac{1}{2}x^2\,dx^3\,dx^4$$
$$\sigma^4 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = \frac{1}{2}\,{s_1}^3 s_4$

9. Normalizers:

$$\Phi_1 = [x^1 = \xi_2 x^4 + x^1\mathrm{e}^{\xi_3}, x^2 = x^2\mathrm{e}^{-\xi_3}, x^3 = x^3 - \xi_3, x^4 = \xi_1 x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1, e_1, 0, e_2], [{e_1}^2 = 1, {e_2}^2 = 1]]$

11. Petrov Reference: [[33, 16, 0]]

A.2.6 G_6 on V_3

A.2.6.1 $F3$

[6, 3, 1]

1. Reference : so(3)+so(3), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	.	Y_4	Y_5	$-Y_2$	$-Y_3$	.
Y_2		.	Y_6	Y_1	.	$-Y_3$
Y_3			.	.	Y_1	Y_2
Y_4				.	Y_6	$-Y_5$
Y_5					.	Y_4
Y_6						.

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_3	$-e_2$	.	.	.
e_2		.	e_1	.	.	.
e_3			.	.	.	.
e_4				.	e_6	$-e_5$
e_5					.	e_4
e_6						.

3. Isomorphisms:

$X_1 \to Y_1, \quad X_3 \to Y_2, \quad X_5 \to Y_3,$
$X_2 \to -Y_5, \quad X_4 \to -Y_6, \quad X_6 \to Y_4$

$X_1 \to e_3 - e_6, \quad X_3 \to e_1 - e_4, \quad X_5 \to -e_2 + e_5,$
$X_2 \to -e_1 - e_4, \quad X_4 \to e_3 + e_6, \quad X_6 \to e_2 + e_5$

4. Isotropy: F3 $[e_3 - e_6, -e_2 + e_5, e_1 + e_4]$

5. Vector Fields Γ:

$$X_1 = -\sin\left(x^3\right)\cos\left(x^4\right)\,\partial_{x^2} - \frac{\cos\left(x^3\right)\cos\left(x^2\right)\cos\left(x^4\right)}{\sin\left(x^2\right)}\,\partial_{x^3} + \frac{\cos\left(x^2\right)\sin\left(x^4\right)}{\sin\left(x^2\right)\sin\left(x^3\right)}\,\partial_{x^4}$$

$$X_2 = \sin\left(x^3\right)\sin\left(x^4\right)\,\partial_{x^2} + \frac{\cos\left(x^3\right)\cos\left(x^2\right)\sin\left(x^4\right)}{\sin\left(x^2\right)}\,\partial_{x^3} + \frac{\cos\left(x^2\right)\cos\left(x^4\right)}{\sin\left(x^2\right)\sin\left(x^3\right)}\,\partial_{x^4}$$

$$X_3 = -\cos\left(x^4\right)\,\partial_{x^3} + \frac{\sin\left(x^4\right)\cos\left(x^3\right)}{\sin\left(x^3\right)}\,\partial_{x^4}$$

$$X_4 = \sin\left(x^4\right)\,\partial_{x^3} + \frac{\cos\left(x^4\right)\cos\left(x^3\right)}{\sin\left(x^3\right)}\,\partial_{x^4}$$

$$X_5 = \partial_{x^4}$$

$$X_6 = -\cos\left(x^3\right)\,\partial_{x^2} + \frac{\sin\left(x^3\right)\cos\left(x^2\right)}{\sin\left(x^2\right)}\,\partial_{x^3}$$

6. Base Point: $[0, \pi/2, \pi/2, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1$$

$$\sigma^2 = dx^2\,dx^2 + \left(\sin\left(x^2\right)\right)^2\,dx^3\,dx^3 + \left(\sin\left(x^3\right)\right)^2\left(\sin\left(x^2\right)\right)^2\,dx^4\,dx^4$$

8. Determinants :

$$\det(g) = s_1{s_2}^3\left(\sin\left(x^2\right)\right)^4\left(\sin\left(x^3\right)\right)^2$$

$$\det(g_O) = {s_2}^3\left(\sin\left(x^2\right)\right)^4\left(\sin\left(x^3\right)\right)^2$$

9. Normalizers:

$$\Phi_1 = [x^1 = A\left(x^1\right), x^2 = x^2, x^3 = x^3, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[e, s_2\left(x^1\right)], [e^2 = 1]]$

11. Petrov Reference: missing from Petrov

[6, 3, 2]

1. Reference : euc(3), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	.	.	.	$-Y_2$	$-Y_3$	.
Y_2		.	.	Y_1	.	$-Y_3$
Y_3			.	.	Y_1	Y_2
Y_4				.	Y_6	$-Y_5$
Y_5					.	Y_4
Y_6						.

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	.	$-e_2$	$-e_3$	.
e_2		.	.	e_1	.	$-e_3$
e_3			.	.	e_1	e_2
e_4				.	e_6	$-e_5$
e_5					.	e_4
e_6						.

3. Isomorphisms:

$[X_1 \to Y_6, X_2 \to Y_4, X_3 \to Y_2, X_4 \to Y_5, X_5 \to Y_1, X_6 \to -Y_3]$
$[X_1 \to e_6, X_2 \to e_4, X_3 \to e_2, X_4 \to e_5, X_5 \to e_1, X_6 \to -e_3]$

4. Isotropy: F3 $[e_4, e_5, e_6]$

5. Vector Fields Γ:

$$X_1 = x^4\,\partial_{x^2} - x^2\,\partial_{x^4} \quad X_4 = x^3\,\partial_{x^2} - x^2\,\partial_{x^3}$$
$$X_2 = x^4\,\partial_{x^3} - x^3\,\partial_{x^4} \quad X_5 = -\,\partial_{x^3}$$
$$X_3 = -\,\partial_{x^4} \quad X_6 = -\,\partial_{x^2}$$

6. Base Point: $[0,0,0,0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1 \quad \sigma^2 = dx^2\,dx^2 + dx^3\,dx^3 + dx^4\,dx^4$$

8. Determinants :

$\det(g) = s_1{s_2}^3$

$\det(g_O) = {s_2}^3$

9. Normalizers:

$$\Phi_1 = [x^1 = B\left(x^1\right), x^2 = x^2, x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2\xi_1, x^3 = x^3\xi_1, x^4 = \xi_1 x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$$[[e, s_2\left(x^1\right)], [e^2 = 1]]$$

11. Petrov Reference: [[33, 40, 1], [33, 44, 0]]

[6, 3, 3]

1. Reference : so(3,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	.	$-Y_4$	$-Y_5$	$-Y_2$	$-Y_3$	.
Y_2		.	$-Y_6$	Y_1	.	$-Y_3$
Y_3			.	.	Y_1	Y_2
Y_4				.	Y_6	$-Y_5$
Y_5					.	Y_4
Y_6						.

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_3	$-e_2$	e_5	$-e_4$	.
e_2		.	e_1	e_6	.	$-e_4$
e_3			.	.	e_6	$-e_5$
e_4				.	$-e_1$	$-e_2$
e_5					.	$-e_3$
e_6						.

3. Isomorphisms:

$[X_1 \to -Y_1, X_2 \to Y_2, X_3 \to Y_3, X_4 \to -Y_4, X_5 \to -Y_6, X_6 \to Y_5]$
$[X_1 \to -e_5, X_2 \to -e_6, X_3 \to e_4, X_4 \to e_3, X_5 \to -e_2, X_6 \to -e_1]$

4. Isotropy: F3 $[e_3, -e_1, -e_2]$

5. Vector Fields Γ:

$$
\begin{array}{ll}
X_1 = x^4\,\partial_{x^2} + x^2\,\partial_{x^4} & X_4 = x^3\,\partial_{x^2} - x^2\,\partial_{x^3} \\
X_2 = x^4\,\partial_{x^3} + x^3\,\partial_{x^4} & X_5 = x^3\,\partial_{x^1} - x^1\,\partial_{x^3} \\
X_3 = x^4\,\partial_{x^1} + x^1\,\partial_{x^4} & X_6 = x^2\,\partial_{x^1} - x^1\,\partial_{x^2}
\end{array}
$$

6. Base Point: $[0, 0, 0, 1]$

7. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$
\begin{aligned}
\sigma^1 &= {x^1}^2\,dx^1\,dx^1 + x^2\,x^1\,dx^1\,dx^2 + x^1\,x^3\,dx^1\,dx^3 - x^1\,x^4\,dx^1\,dx^4 + {x^2}^2\,dx^2\,dx^2 \\
&\quad + x^2\,x^3\,dx^2\,dx^3 - x^2\,x^4\,dx^2\,dx^4 + {x^3}^2\,dx^3\,dx^3 - x^3\,x^4\,dx^3\,dx^4 + {x^4}^2\,dx^4\,dx^4 \\
\sigma^2 &= dx^1\,dx^1 + dx^2\,dx^2 + dx^3\,dx^3 - dx^4\,dx^4
\end{aligned}
$$

8. Determinants :

$\det(g) = -{s_2}^3 \left(s_1 {x^1}^2 + s_1 {x^2}^2 + s_1 {x^3}^2 - s_1 {x^4}^2 + s_2\right)$

$\det(g_O) = -\frac{{s_2}^3\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)}{{x^4}^2}$

9. Normalizers: not computed

10. Normalized Metrics (with respect to the invariant quadratic forms):

$$[[s_1 \left(-\left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2 - \left(x^4\right)^2\right), s_2 \left(-\left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2 - \left(x^4\right)^2\right)]]$$

11. Petrov Reference: [[33, 40, 2]]

A.2.6.2 $F4$

[6, 3, 4]

1. Reference : so(2,1)+so(2,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	.	$-Y_4$	$-Y_5$	$-Y_2$	$-Y_3$	.
Y_2		.	$-Y_6$	Y_1	.	$-Y_3$
Y_3			.	.	$-Y_1$	$-Y_2$
Y_4				.	Y_6	$-Y_5$
Y_5					.	$-Y_4$
Y_6						.

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_2	$-e_3$	.	.	.
e_2		.	$-e_1$	.	.	.
e_3			.	.	.	.
e_4				.	e_5	$-e_6$
e_5					.	$-e_4$
e_6						.

3. Isomorphisms:

$$X_1 \to \frac{1}{2}\,Y_1 + \frac{1}{2}\,Y_6, \qquad X_4 \to -\frac{1}{2}\,Y_1 + \frac{1}{2}\,Y_6,$$
$$X_2 \to -\frac{1}{2}\,Y_2 - \frac{1}{2}\,Y_3 + \frac{1}{2}\,Y_4 + \frac{1}{2}\,Y_5, \quad X_5 \to \frac{1}{2}\,Y_2 + \frac{1}{2}\,Y_3 + \frac{1}{2}\,Y_4 + \frac{1}{2}\,Y_5,$$
$$X_3 \to \frac{1}{4}\,Y_2 - \frac{1}{4}\,Y_3 + \frac{1}{4}\,Y_4 - \frac{1}{4}\,Y_5, \quad X_6 \to -\frac{1}{4}\,Y_2 + \frac{1}{4}\,Y_3 + \frac{1}{4}\,Y_4 - \frac{1}{4}\,Y_5$$

$$X_1 \to e_1, \quad X_4 \to e_4,$$
$$X_2 \to e_2, \quad X_5 \to e_5,$$
$$X_3 \to e_3, \quad X_6 \to e_6$$

4. Isotropy: F4 $[-\frac{1}{2}\,e_2 - e_3 - \frac{1}{2}\,e_5 - e_6, \frac{1}{2}\,e_2 - e_3 - \frac{1}{2}\,e_5 + e_6, -e_1 + e_4]$

5. Vector Fields Γ:

$$X_1 = -x^2\,\partial_{x^2} - \partial_{x^3} \qquad X_4 = x^1\,\partial_{x^1} + \partial_{x^3}$$
$$X_2 = \frac{1}{2}\,\partial_{x^2} \qquad X_5 = -\frac{1}{2}\,\left(x^1\right)^2\,\partial_{x^1} - \frac{1}{2}\,\mathrm{e}^{x^3}\,\partial_{x^2} - x^1\,\partial_{x^3}$$
$$X_3 = \mathrm{e}^{x^3}\,\partial_{x^1} + x^4\,\partial_{x^2} + 2\,x^2\,\partial_{x^3} \qquad X_6 = -\,\partial_{x^1}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = -2\,\mathrm{e}^{-x^3}\,dx^1\,dx^2 + dx^3\,dx^3 \quad \sigma^2 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -4\,{s_1}^3\mathrm{e}^{-2\,x^3}s_2$

$\det(g_O) = -4\,{s_1}^3\mathrm{e}^{-2\,x^3}$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = A\left(x^4\right)]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$$[[s_1\left(x^4\right), e], [e^2 = 1]]$$

11. Petrov Reference: missing from Petrov

[6, 3, 5]

1. Reference : euc(2,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	.	.	.	$-Y_2$	$-Y_3$	.
Y_2		.	.	Y_1	.	$-Y_3$
Y_3			.	.	$-Y_1$	$-Y_2$
Y_4				.	Y_6	$-Y_5$
Y_5					.	$-Y_4$
Y_6						.

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	.	$-e_2$	$-e_3$	.
e_2		.	.	e_1	.	$-e_3$
e_3			.	.	$-e_1$	$-e_2$
e_4				.	e_6	$-e_5$
e_5					.	$-e_4$
e_6						.

3. Isomorphisms:

$[X_1 \to Y_5, X_2 \to Y_6, X_3 \to -Y_3, X_4 \to Y_4, X_5 \to -Y_2, X_6 \to -Y_1]$
$[X_1 \to e_5, X_2 \to e_6, X_3 \to -e_3, X_4 \to e_4, X_5 \to -e_2, X_6 \to -e_1]$

4. Isotropy: F4 $[e_4, e_5, e_6]$

5. Vector Fields Γ:

$$X_1 = x^4\,\partial_{x^2} + x^2\,\partial_{x^4} \quad X_4 = x^3\,\partial_{x^2} - x^2\,\partial_{x^3}$$
$$X_2 = x^4\,\partial_{x^3} + x^3\,\partial_{x^4} \quad X_5 = -\,\partial_{x^3}$$
$$X_3 = \partial_{x^4} \quad X_6 = -\,\partial_{x^2}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1 \quad \sigma^2 = -\,dx^2\,dx^2 - dx^3\,dx^3 + dx^4\,dx^4$$

8. Determinants :

$\det(g) = s_1 {s_2}^3$

$\det(g_O) = {s_2}^3$

9. Normalizers:

$$\Phi_1 = [x^1 = B\left(x^1\right), x^2 = x^2, x^3 = x^3, x^4 = x^4]$$
$$\Phi_2 = [x^1 = x^1, x^2 = x^2\xi_1, x^3 = x^3\xi_1, x^4 = \xi_1 x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$$[[e, s_2\left(x^1\right)], [e^2 = 1]]$$

11. Petrov Reference: [[33, 40, 0]]

[6, 3, 6]

1. Reference : so(3,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	.	Y_4	Y_5	$-Y_2$	$-Y_3$	.
Y_2		.	Y_6	Y_1	.	$-Y_3$
Y_3			.	.	$-Y_1$	$-Y_2$
Y_4				.	Y_6	$-Y_5$
Y_5					.	$-Y_4$
Y_6						.

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_3	$-e_2$	e_5	$-e_4$	.
e_2		.	e_1	e_6	.	$-e_4$
e_3			.	.	e_6	$-e_5$
e_4				.	$-e_1$	$-e_2$
e_5					.	$-e_3$
e_6						.

3. Isomorphisms:

$[X_1 \to -Y_5, X_2 \to -Y_6, X_3 \to Y_3, X_4 \to -Y_4, X_5 \to Y_2, X_6 \to Y_1]$
$[X_1 \to e_5, X_2 \to e_6, X_3 \to -e_4, X_4 \to -e_3, X_5 \to e_2, X_6 \to e_1]$

4. Isotropy: F4 $[-e_3, e_6, e_5]$

5. Vector Fields Γ:

$$X_1 = x^4\,\partial_{x^2} + x^2\,\partial_{x^4} \quad X_4 = -x^3\,\partial_{x^2} + x^2\,\partial_{x^3}$$
$$X_2 = x^4\,\partial_{x^3} + x^3\,\partial_{x^4} \quad X_5 = -x^3\,\partial_{x^1} + x^1\,\partial_{x^3}$$
$$X_3 = x^4\,\partial_{x^1} + x^1\,\partial_{x^4} \quad X_6 = -x^2\,\partial_{x^1} + x^1\,\partial_{x^2}$$

6. Base Point: $[1, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = {x^1}^2\,dx^1\,dx^1 + x^2\,x^1\,dx^1\,dx^2 + x^1\,x^3\,dx^1\,dx^3 - x^1\,x^4\,dx^1\,dx^4 + {x^2}^2\,dx^2\,dx^2$$
$$+ x^2\,x^3\,dx^2\,dx^3 - x^2\,x^4\,dx^2\,dx^4 + {x^3}^2\,dx^3\,dx^3 - x^3\,x^4\,dx^3\,dx^4 + {x^4}^2\,dx^4\,dx^4$$
$$\sigma^2 = dx^1\,dx^1 + dx^2\,dx^2 + dx^3\,dx^3 - dx^4\,dx^4$$

8. Determinants :

$\det(g) = -{s_2}^3\left(s_1{x^1}^2 + s_1{x^2}^2 + s_1{x^3}^2 - s_1{x^4}^2 + s_2\right)$

$\det(g_O) = -\frac{{s_2}^3\left({x^1}^2+{x^2}^2+{x^3}^2-{x^4}^2\right)}{{x^4}^2}$

9. Normalizers: not computed

10. Normalized Metrics (with respect to the invariant quadratic forms):

$$[[s_1\left(-\left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2 - \left(x^4\right)^2\right), s_2\left(-\left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2 - \left(x^4\right)^2\right)]]$$

11. Petrov Reference: [[33, 40, 3], [33, 45, 0]]

A.2.7 G_6 on V_4

A.2.7.1 Non-reductive

[6, 4, -1]

1. Reference : A4, Fels

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	.	Y_2	$-2\,Y_3+4\,Y_4$	$2\,Y_4$	$-Y_3+Y_4$	$-Y_6$
Y_2		.	$-2\,Y_6$	.	$-Y_6$	$-Y_3+Y_4+2\,Y_5$
Y_3			.	$-2\,Y_1$	Y_1	Y_2
Y_4				.	Y_1	Y_2
Y_5					.	.
Y_6						.

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	$2\,e_2$	$-2\,e_3$	e_4	$-e_5$	.
e_2		.	e_1	.	e_4	.
e_3			.	e_5	.	.
e_4				.	e_6	.
e_5					.	.
e_6						.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_4, X_3 \to \frac{1}{2}\,Y_3 - \frac{1}{2}\,Y_4, X_4 \to \frac{1}{2}\,\sqrt{2}\,Y_2, X_5 \to \frac{1}{2}\,\sqrt{2}\,Y_6, X_6 \to -\frac{1}{2}\,Y_3 + \frac{1}{2}\,Y_4 + Y_5]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_4, X_5 \to e_5, X_6 \to e_6]$

4. Isotropy: F10 $[\,e_3 + e_6, \, e_5]$

5. Vector Fields Γ:

$$X_1 = (-2\,x^1 + 2)\,\partial_{x^1} + \frac{x^2x^1 + x^1 - 2\,x^2 - 1}{x^1}\,\partial_{x^2} + \partial_{x^3} + (-x^4x^1 - 2\,x^1x^5 + 2\,x^7 + x^2x^1 + 2\,x^5 + 1)\,\partial_{x^4}$$
$$X_2 = \partial_{x^1} - \frac{x^2+1}{x^1}\,\partial_{x^2} + (x^7 + 2\,x^5 + \frac{1}{2})\,\partial_{x^4}$$

$$X_3 = -\left(x^1-1\right)^2\,\partial_{x^1} - \frac{x^2\left(x^1-1\right)}{x^1}\,\partial_{x^2} + (x^1-1)\,\partial_{x^3} + (-1 + \left(x^1\right)^2 x^7 - 3/2\,x^4\left(x^1\right)^2 + x^4x^1 - x^7 - \mathrm{e}^{-2\,x^3} + x^1)\,\partial_{x^4}$$
$$X_4 = -\left(x^1\right)^{-1}\,\partial_{x^2} + (2\,x^5+1)\,\partial_{x^4}$$
$$X_5 = -\frac{x^1-1}{x^1}\,\partial_{x^2} + (2\,x^1x^5 - x^2x^1 - 2\,x^5)\,\partial_{x^4}$$
$$X_6 = \partial_{x^4}$$

6. Base Point: $[1, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = \mathrm{e}^{4\,x^3}\,dx^1\,dx^1$$
$$\sigma^2 = -\frac{1}{2}\,{x^2}^2\mathrm{e}^{2\,x^3}\left(-1+2\,x^3\right)\,dx^1\,dx^1 - \frac{1}{2}\,\left(-1+2\,x^3\right)x^1\,x^2\,\mathrm{e}^{2\,x^3}\,dx^1\,dx^2 - \frac{1}{2}\,\left({x^2}^2x^1-1\right)\mathrm{e}^{2\,x^3}\,dx^1\,dx^3 - \frac{1}{2}\,\mathrm{e}^{2\,x^3}\,dx^1\,dx^4 + \frac{1}{2}\,{x^1}^2\mathrm{e}^{2\,x^3}\,dx^2\,dx^2 + dx^3\,dx^3$$

8. Determinants :

$\det(g) = -\frac{1}{8}\,{s_2}^4\mathrm{e}^{6\,x^3}{x^1}^2$

9. Normalizers: too large for display

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[e, s_2], [e^2 = 1]]$

11. Petrov Reference: missing from Petrov

A.2.7.2 $F9$

[6, 4, 1]

1. Reference : so(3)+so(2,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	.	Y_5	.	.	$-Y_2$	.
Y_2		.	.	.	Y_1	.
Y_3			.	Y_6	.	Y_4
Y_4				.	.	Y_3
Y_5					.	.
Y_6						.

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_3	$-e_2$	.	.	.
e_2		.	e_1	.	.	.
e_3			.	.	.	.
e_4				.	e_5	$-e_6$
e_5					.	$-e_4$
e_6						.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2, X_3 \to Y_5, X_4 \to Y_4, X_5 \to Y_3, X_6 \to Y_6]$
$[X_1 \to e_1, X_2 \to e_2, X_3 \to e_3, X_4 \to e_5 + \frac{1}{2}e_6, X_5 \to e_5 - \frac{1}{2}e_6, X_6 \to -e_4]$

4. Isotropy: F9 $[-e_1, -e_4 - e_5]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^4}$$
$$X_2 = \cos\left(x^4\right)\,\partial_{x^3} + \frac{\sin\left(x^3\right)\sin\left(x^4\right)}{\cos\left(x^3\right)}\,\partial_{x^4}$$
$$X_3 = -\sin\left(x^4\right)\,\partial_{x^3} + \frac{\sin\left(x^3\right)\cos\left(x^4\right)}{\cos\left(x^3\right)}\,\partial_{x^4}$$
$$X_4 = -\,\partial_{x^2}$$
$$X_5 = \cos\left(x^2\right)\,\partial_{x^1} - \frac{\sin\left(x^2\right)\sinh\left(x^1\right)}{\cosh\left(x^1\right)}\,\partial_{x^2}$$
$$X_6 = -\sin\left(x^2\right)\,\partial_{x^1} - \frac{\cos\left(x^2\right)\sinh\left(x^1\right)}{\cosh\left(x^1\right)}\,\partial_{x^2}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = -\,dx^1\,dx^1 + \left(\cosh\left(x^1\right)\right)^2\,dx^2\,dx^2$$
$$\sigma^2 = dx^3\,dx^3 + \left(\cos\left(x^3\right)\right)^2\,dx^4\,dx^4$$

8. Determinants :

$\det(g) = -{s_1}^2\left(\cosh\left(x^1\right)\right)^2 {s_2}^2\left(\cos\left(x^3\right)\right)^2$

9. Normalizers: not computed

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1, s_2]]$

11. Petrov Reference: [[33, 50, 0], [33, 50, 1], [33, 51, 0]]

[6, 4, 2]

1. Reference : so(2,1)+so(2,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	.	$-Y_5$	.	.	$-Y_2$	.
Y_2		.	.	.	Y_1	.
Y_3			.	Y_6	.	Y_4
Y_4				.	.	Y_3
Y_5					.	.
Y_6						.

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	e_2	$-e_3$	.	.	.
e_2		.	$-e_1$	.	.	.
e_3			.	.	.	.
e_4				.	e_5	$-e_6$
e_5					.	$-e_4$
e_6						.

3. Isomorphisms:

$[X_1 \to Y_1, X_2 \to Y_2 + Y_5, X_3 \to -Y_2 + Y_5, X_4 \to Y_4, X_5 \to Y_3, X_6 \to Y_6]$
$[X_1 \to -e_1, X_2 \to e_2, X_3 \to 2\,e_3, X_4 \to -\frac{1}{2}\,e_5 - e_6, X_5 \to \frac{1}{2}\,e_5 - e_6, X_6 \to e_4]$

4. Isotropy: F9 $[\,e_4 - e_5 - e_6, -e_1 + e_3]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^4}$$
$$X_2 = \mathrm{e}^{-x^4}\,\partial_{x^3} + \frac{\sinh\left(x^3\right)\mathrm{e}^{-x^4}}{\cosh\left(x^3\right)}\,\partial_{x^4}$$
$$X_3 = -\mathrm{e}^{x^4}\,\partial_{x^3} + \frac{\sinh\left(x^3\right)\mathrm{e}^{x^4}}{\cosh\left(x^3\right)}\,\partial_{x^4}$$
$$X_4 = -\,\partial_{x^2}$$
$$X_5 = \cos\left(x^2\right)\,\partial_{x^1} - \frac{\sin\left(x^2\right)\sinh\left(x^1\right)}{\cosh\left(x^1\right)}\,\partial_{x^2}$$
$$X_6 = -\sin\left(x^2\right)\,\partial_{x^1} - \frac{\cos\left(x^2\right)\sinh\left(x^1\right)}{\cosh\left(x^1\right)}\,\partial_{x^2}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = -\,dx^1\,dx^1 + \left(\cosh\left(x^1\right)\right)^2\,dx^2\,dx^2$$
$$\sigma^2 = dx^3\,dx^3 + \left(\cosh\left(x^3\right)\right)^2\,dx^4\,dx^4$$

8. Determinants :

$\det(g) = -{s_1}^2\left(\cosh\left(x^1\right)\right)^2 {s_2}^2\left(\cosh\left(x^3\right)\right)^2$

9. Normalizers: not computed

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1, s_2]]$

11. Petrov Reference: [[33, 51, 1], [33, 52, 0], [33, 52, 1]]

[6, 4, 3]

1. Reference : s(3,3)(a = 1)+so(2,1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	.	.	.	.	$-Y_2$	.
Y_2		.	.	.	Y_1	.
Y_3			.	$-Y_6$	.	Y_4
Y_4				.	.	Y_3
Y_5					.	.
Y_6						.

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	e_2	.	.	.
e_2		.	$-e_1$	.	.	.
e_3			.	.	.	.
e_4				.	e_5	$-e_6$
e_5					.	$-e_4$
e_6						.

3. Isomorphisms:

$$X_1 \to Y_3, \quad X_3 \to Y_6, \quad X_5 \to Y_1,$$
$$X_2 \to Y_4, \quad X_4 \to Y_5, \quad X_6 \to Y_2$$

$$X_1 \to -\frac{1}{2}\, e_5 - e_6, \quad X_3 \to e_4, \quad X_5 \to -\frac{1}{2}\, e_1 + \frac{1}{2}\, e_2,$$
$$X_2 \to \frac{1}{2}\, e_5 - e_6, \quad X_4 \to e_3, \quad X_6 \to \frac{1}{2}\, e_1 + \frac{1}{2}\, e_2$$

4. Isotropy: F9 $[-e_1 - e_2 - e_3, -e_4]$

5. Vector Fields Γ:

$$X_1 = -\,\partial_{x^4}$$
$$X_2 = \cos\left(x^4\right)\,\partial_{x^3} - \frac{\sinh\left(x^3\right)\sin\left(x^4\right)}{\cosh\left(x^3\right)}\,\partial_{x^4}$$
$$X_3 = -\sin\left(x^4\right)\,\partial_{x^3} - \frac{\sinh\left(x^3\right)\cos\left(x^4\right)}{\cosh\left(x^3\right)}\,\partial_{x^4}$$
$$X_4 = -x^2\,\partial_{x^1} + x^1\,\partial_{x^2}$$
$$X_5 = \partial_{x^2}$$
$$X_6 = \partial_{x^1}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1 + dx^2\,dx^2$$
$$\sigma^2 = -\,dx^3\,dx^3 + \left(\cosh\left(x^3\right)\right)^2\,dx^4\,dx^4$$

8. Determinants :

$\det(g) = -{s_1}^2{s_2}^2\left(\cosh\left(x^3\right)\right)^2$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1\xi_1, x^2 = x^2\xi_1, x^3 = x^3, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[e, s_2], [e^2 = 1]]$

11. Petrov Reference: [[33, 48, 1], [33, 49, 1]]

[6, 4, 4]

1. Reference : s(3,1)(a = -1)+so(3), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	.	Y_5	.	.	$-Y_2$	.
Y_2		.	.	.	Y_1	.
Y_3			.	.	.	Y_4
Y_4				.	.	Y_3
Y_5					.	.
Y_6						.

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	$-e_1$	.	.	.
e_2		.	e_2	.	.	.
e_3			.	.	.	.
e_4				.	e_6	$-e_5$
e_5					.	e_4
e_6						.

3. Isomorphisms:

$$X_1 \to Y_1, \quad X_3 \to Y_5, \quad X_5 \to Y_3,$$
$$X_2 \to Y_2, \quad X_4 \to Y_6, \quad X_6 \to Y_4$$

$$X_1 \to e_4, \quad X_3 \to e_6, \quad X_5 \to -\frac{1}{2}\, e_1 + \frac{1}{2}\, e_2,$$
$$X_2 \to e_5, \quad X_4 \to e_3, \quad X_6 \to \frac{1}{2}\, e_1 + \frac{1}{2}\, e_2$$

4. Isotropy: F9 $[\, e_6, \frac{1}{2}\, e_1 - \frac{1}{2}\, e_2 - e_3]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^4}$$
$$X_2 = \cos\left(x^4\right)\, \partial_{x^3} + \frac{\sin\left(x^3\right)\sin\left(x^4\right)}{\cos\left(x^3\right)}\, \partial_{x^4}$$
$$X_3 = -\sin\left(x^4\right)\, \partial_{x^3} + \frac{\sin\left(x^3\right)\cos\left(x^4\right)}{\cos\left(x^3\right)}\, \partial_{x^4}$$
$$X_4 = -x^2\, \partial_{x^1} - x^1\, \partial_{x^2}$$
$$X_5 = -\, \partial_{x^2}$$
$$X_6 = \partial_{x^1}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = -\, dx^1\, dx^1 + \, dx^2\, dx^2$$
$$\sigma^2 = \, dx^3\, dx^3 + \left(\cos\left(x^3\right)\right)^2\, dx^4\, dx^4$$

8. Determinants :

$\det(g) = -{s_1}^2{s_2}^2 \left(\cos\left(x^3\right)\right)^2$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1\xi_1, x^2 = x^2\xi_1, x^3 = x^3, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$$[[e, s_2], [e^2 = 1]]$$

11. Petrov Reference: [[33, 48, 0]]

[6, 4, 5]

1. Reference : s(3,1)(a = -1)+so(2, 1), Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_1	.	$-Y_5$	.	.	$-Y_2$	.
Y_2		.	.	.	Y_1	.
Y_3			.	.	.	Y_4
Y_4				.	.	Y_3
Y_5					.	.
Y_6						.

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	.	.	$-e_1$	.	.	.
e_2		.	e_2	.	.	.
e_3			.	.	.	.
e_4				.	e_5	$-e_6$
e_5					.	$-e_4$
e_6						.

3. Isomorphisms:

$$X_1 \to Y_1, \quad X_3 \to -Y_2 + Y_5, \quad X_5 \to Y_3,$$
$$X_2 \to Y_2 + Y_5, \quad X_4 \to Y_6, \quad X_6 \to Y_4$$

$$X_1 \to -e_4, \quad X_3 \to -2\,e_6, \quad X_5 \to -\frac{1}{2}\,e_1 + \frac{1}{2}\,e_2,$$
$$X_2 \to -e_5, \quad X_4 \to e_3, \quad X_6 \to \frac{1}{2}\,e_1 + \frac{1}{2}\,e_2$$

4. Isotropy: F9 $[-e_4 + e_5 + e_6, -\frac{1}{2}\,e_1 - \frac{1}{2}\,e_2 - e_3]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^4}$$
$$X_2 = \mathrm{e}^{-x^4}\,\partial_{x^3} + \frac{\sinh\left(x^3\right)\mathrm{e}^{-x^4}}{\cosh\left(x^3\right)}\,\partial_{x^4}$$
$$X_3 = -\mathrm{e}^{x^4}\,\partial_{x^3} + \frac{\sinh\left(x^3\right)\mathrm{e}^{x^4}}{\cosh\left(x^3\right)}\,\partial_{x^4}$$
$$X_4 = -x^2\,\partial_{x^1} - x^1\,\partial_{x^2}$$
$$X_5 = -\,\partial_{x^2}$$
$$X_6 = \partial_{x^1}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = -\,dx^1\,dx^1 + dx^2\,dx^2$$
$$\sigma^2 = dx^3\,dx^3 + \left(\cosh\left(x^3\right)\right)^2\,dx^4\,dx^4$$

8. Determinants :

$\det(g) = -{s_1}^2{s_2}^2\left(\cosh\left(x^3\right)\right)^2$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1\xi_1, x^2 = x^2\xi_1, x^3 = x^3, x^4 = x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[e, s_2], [e^2 = 1]]$

11. Petrov Reference: [[33, 49, 0]]

A.2.7.3 $F10$

See Section 3.2.3 for information regarding this case.

A.2.8 G_7 on V_4

A.2.8.1 $F3$

[7, 4, 1]

1. Reference : so(3)+so(3)+R, Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7
Y_1	.	.	.	.	.	.	.
Y_2		.	Y_7	$-Y_6$	.	Y_4	$-Y_3$
Y_3			.	Y_5	$-Y_4$	.	Y_2
Y_4				.	Y_3	$-Y_2$	.
Y_5					.	Y_7	$-Y_6$
Y_6						.	Y_5
Y_7							.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	e_3	$-e_2$	.	.	.	.
e_2		.	e_1	.	.	.	.
e_3			.	.	.	.	.
e_4				.	e_6	$-e_5$	.
e_5					.	e_4	.
e_6						.	.
e_7							.

3. Isomorphisms:

$$X_1 \to Y_1, \quad X_4 \to Y_3, \quad X_7 \to Y_7$$
$$X_2 \to Y_2, \quad X_5 \to -Y_5,$$
$$X_3 \to Y_6, \quad X_6 \to Y_4$$

$$X_1 \to e_7, \quad X_4 \to e_1 - e_4, \quad X_7 \to e_2 + e_5$$
$$X_2 \to e_3 - e_6, \quad X_5 \to e_3 + e_6,$$
$$X_3 \to -e_1 - e_4, \quad X_6 \to -e_2 + e_5$$

4. Isotropy: F3 $[-e_3 - e_6, e_2 - e_5, e_1 - e_4]$

5. Vector Fields Γ:

$$X_1 = \partial_{x^1}$$
$$X_2 = -\sin\left(x^3\right)\cos\left(x^4\right)\,\partial_{x^2} - \frac{\cos\left(x^3\right)\cos\left(x^2\right)\cos\left(x^4\right)}{\sin\left(x^2\right)}\,\partial_{x^3} + \frac{\cos\left(x^2\right)\sin\left(x^4\right)}{\sin\left(x^2\right)\sin\left(x^3\right)}\,\partial_{x^4}$$
$$X_3 = \sin\left(x^3\right)\sin\left(x^4\right)\,\partial_{x^2} + \frac{\cos\left(x^3\right)\cos\left(x^2\right)\sin\left(x^4\right)}{\sin\left(x^2\right)}\,\partial_{x^3} + \frac{\cos\left(x^2\right)\cos\left(x^4\right)}{\sin\left(x^2\right)\sin\left(x^3\right)}\,\partial_{x^4}$$
$$X_4 = -\cos\left(x^4\right)\,\partial_{x^3} + \frac{\sin\left(x^4\right)\cos\left(x^3\right)}{\sin\left(x^3\right)}\,\partial_{x^4}$$
$$X_5 = \sin\left(x^4\right)\,\partial_{x^3} + \frac{\cos\left(x^4\right)\cos\left(x^3\right)}{\sin\left(x^3\right)}\,\partial_{x^4}$$
$$X_6 = \partial_{x^4}$$
$$X_7 = -\cos\left(x^3\right)\,\partial_{x^2} + \frac{\sin\left(x^3\right)\cos\left(x^2\right)}{\sin\left(x^2\right)}\,\partial_{x^3}$$

6. Base Point: $[0, \pi/2, \pi/2, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = dx^1\,dx^1$$
$$\sigma^2 = dx^2\,dx^2 + \left(\sin\left(x^2\right)\right)^2\,dx^3\,dx^3 + \left(\sin\left(x^3\right)\right)^2\left(\sin\left(x^2\right)\right)^2\,dx^4\,dx^4$$

8. Determinants :

$$\det(g) = s_1 {s_2}^3\left(\sin\left(x^2\right)\right)^4\left(\sin\left(x^3\right)\right)^2$$

9. Normalizers:

$\Phi_1 = [x^1 = x^1\xi_1, x^2 = x^2, x^3 = x^3, x^4 = x^4]$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[e, s_2], [e^2 = 1]]$

11. Petrov Reference: [[33, 41, 0]]

[7, 4, 2]

1. Reference : so(3,1)+R, Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7
Y_1	.	.	.	.	.	.	.
Y_2		.	$-Y_7$	Y_6	.	Y_4	$-Y_3$
Y_3			.	$-Y_5$	$-Y_4$	.	Y_2
Y_4				.	Y_3	$-Y_2$	.
Y_5					.	Y_7	$-Y_6$
Y_6						.	Y_5
Y_7							.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	e_2	e_3	$-e_4$	$-e_5$	.	.
e_2		.	.	$-e_1$	e_6	$-e_3$	.
e_3			.	$-e_6$	$-e_1$	e_2	.
e_4				.	.	$-e_5$	.
e_5					.	e_4	.
e_6						.	.
e_7							.

3. Isomorphisms:

$$X_1 \to Y_1, \quad X_4 \to Y_4, \quad X_7 \to -Y_6$$
$$X_2 \to -Y_2, \quad X_5 \to -Y_7,$$
$$X_3 \to Y_3, \quad X_6 \to -Y_5$$

$$X_1 \to e_7, \qquad X_4 \to \frac{1}{2}\sqrt{2}\,e_3 - \frac{1}{2}\sqrt{2}\,e_5, \quad X_7 \to e_6$$
$$X_2 \to -\frac{1}{2}\sqrt{2}\,e_2 + \frac{1}{2}\sqrt{2}\,e_4, \quad X_5 \to \frac{1}{2}\sqrt{2}\,e_2 + \frac{1}{2}\sqrt{2}\,e_4,$$
$$X_3 \to -e_1, \qquad X_6 \to -\frac{1}{2}\sqrt{2}\,e_3 - \frac{1}{2}\sqrt{2}\,e_5$$

4. Isotropy: F3 $[-e_1 - e_3 - e_5, -e_4 - e_6, e_2 - e_6]$

5. Vector Fields Γ:

$$X_1 = x^1\,\partial_{x^1} + x^2\,\partial_{x^2} + x^3\,\partial_{x^3} + x^4\,\partial_{x^4} \qquad X_5 = x^3\,\partial_{x^2} - x^2\,\partial_{x^3}$$
$$X_2 = x^4\,\partial_{x^2} + x^2\,\partial_{x^4} \qquad X_6 = x^3\,\partial_{x^1} - x^1\,\partial_{x^3}$$
$$X_3 = x^4\,\partial_{x^3} + x^3\,\partial_{x^4} \qquad X_7 = x^2\,\partial_{x^1} - x^1\,\partial_{x^2}$$
$$X_4 = x^4\,\partial_{x^1} + x^1\,\partial_{x^4}$$

6. Base Point: $[0, 0, 0, 1]$

7. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = \frac{{x^1}^2}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\,dx^1\,dx^1 + \frac{x^2\,x^1}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\,dx^1\,dx^2$$
$$+ \frac{x^1\,x^3}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\,dx^1\,dx^3 - \frac{x^1\,x^4}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\,dx^1\,dx^4$$
$$+ \frac{{x^2}^2}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\,dx^2\,dx^2 + \frac{x^2\,x^3}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\,dx^2\,dx^3$$
$$- \frac{x^2\,x^4}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\,dx^2\,dx^4 + \frac{{x^3}^2}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\,dx^3\,dx^3$$
$$- \frac{x^3\,x^4}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\,dx^3\,dx^4 + \frac{{x^4}^2}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\,dx^4\,dx^4$$

$$\sigma^2 = -\frac{{x^2}^2 + {x^3}^2 - {x^4}^2}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\, dx^1\, dx^1 + \frac{x^2\, x^1}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\, dx^1\, dx^2$$
$$+ \frac{x^1\, x^3}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\, dx^1\, dx^3 - \frac{x^1\, x^4}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\, dx^1\, dx^4$$
$$- \frac{{x^1}^2 + {x^3}^2 - {x^4}^2}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\, dx^2\, dx^2 + \frac{x^2\, x^3}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\, dx^2\, dx^3$$
$$- \frac{x^2\, x^4}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\, dx^2\, dx^4 - \frac{{x^1}^2 + {x^2}^2 - {x^4}^2}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\, dx^3\, dx^3$$
$$- \frac{x^3\, x^4}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\, dx^3\, dx^4 + \frac{{x^1}^2 + {x^2}^2 + {x^3}^2}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\, dx^4\, dx^4$$

8. Determinants :

$\det(g) = \frac{s_1 {s_2}^3}{\left({x^1}^2+{x^2}^2+{x^3}^2-{x^4}^2\right)^4}$

9. Normalizers:

$\Phi_1 = [x^1 = x^1\xi_1, x^2 = x^2, x^3 = x^3, x^4 = x^4]$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1, s_2]]$

11. Petrov Reference: [[33, 41, 1]]

A.2.8.2 $F4$

[7, 4, 3]

1. Reference : so(3,1)+R, Snobl

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7
Y_1	.	Y_6	.	Y_7	.	$-Y_2$	$-Y_4$
Y_2		.	.	Y_5	$-Y_4$	Y_1	.
Y_3			.	.	.	.	.
Y_4				.	$-Y_2$	.	$-Y_1$
Y_5					.	Y_7	Y_6
Y_6						.	Y_5
Y_7							.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	e_3	$-e_2$	e_5	$-e_4$	.	.
e_2		.	e_1	e_6	.	$-e_4$	.
e_3			.	.	e_6	$-e_5$	.
e_4				.	$-e_1$	$-e_2$	.
e_5					.	$-e_3$	.
e_6						.	.
e_7							.

3. Isomorphisms:

$[X_1 \to -Y_5, X_2 \to -Y_7, X_3 \to Y_4, X_4 \to Y_6, X_5 \to Y_1, X_6 \to Y_2, X_7 \to Y_3]$
$[X_1 \to e_6, X_2 \to e_5, X_3 \to e_4, X_4 \to e_3, X_5 \to -e_1, X_6 \to -e_2, X_7 \to e_7]$

4. Isotropy: F4 $[e_3, -e_5, -e_6]$

5. Vector Fields Γ:

$$X_1 = x^4\,\partial_{x^2} + x^2\,\partial_{x^4}$$

$$X_2 = x^4\,\partial_{x^3} + x^3\,\partial_{x^4}$$

$$X_3 = x^4\,\partial_{x^1} + x^1\,\partial_{x^4}$$

$$X_4 = -x^3\,\partial_{x^2} + x^2\,\partial_{x^3}$$

$$X_5 = -x^3\,\partial_{x^1} + x^1\,\partial_{x^3}$$

$$X_6 = -x^2\,\partial_{x^1} + x^1\,\partial_{x^2}$$

$$X_7 = \left(\left(x^1\right)^2 + x^4 + x^6 - x^8\right) x^1\,\partial_{x^1} \left(\left(x^1\right)^2 + x^4 + x^6 - x^8\right) x^2\,\partial_{x^2} \left(\left(x^1\right)^2 + x^4 + x^6 - x^8\right) x^3\,\partial_{x^3} \left(\left(x^1\right)^2 + x^4 + x^6 - x^8\right) x^4\,\partial_{x^4}$$

6. Base Point: $[1, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i \sigma^i$

$$\sigma^1 = \frac{1}{2}\,\frac{{x^2}^2 + {x^3}^2 - {x^4}^2}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\,dx^1\,dx^1 - \frac{1}{2}\,\frac{x^2\,x^1}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\,dx^1\,dx^2$$
$$- \frac{1}{2}\,\frac{x^1\,x^3}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\,dx^1\,dx^3 + \frac{1}{2}\,\frac{x^1\,x^4}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\,dx^1\,dx^4$$
$$+ \frac{1}{2}\,\frac{{x^1}^2 + {x^3}^2 - {x^4}^2}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\,dx^2\,dx^2 - \frac{1}{2}\,\frac{x^2\,x^3}{\left({x^1}^2 + {x^2}^2 + {x^3}^2 - {x^4}^2\right)^2}\,dx^2\,dx^3$$

$$+\frac{1}{2}\frac{x^2\,x^4}{\left({x^1}^2+{x^2}^2+{x^3}^2-{x^4}^2\right)^2}\,dx^2\,dx^4+\frac{1}{2}\frac{{x^1}^2+{x^2}^2-{x^4}^2}{\left({x^1}^2+{x^2}^2+{x^3}^2-{x^4}^2\right)^2}\,dx^3\,dx^3$$

$$+\frac{1}{2}\frac{x^3\,x^4}{\left({x^1}^2+{x^2}^2+{x^3}^2-{x^4}^2\right)^2}\,dx^3\,dx^4-\frac{1}{2}\frac{{x^1}^2+{x^2}^2+{x^3}^2}{\left({x^1}^2+{x^2}^2+{x^3}^2-{x^4}^2\right)^2}\,dx^4\,dx^4$$

$$\sigma^2=\frac{{x^1}^2}{\left({x^1}^2+{x^2}^2+{x^3}^2-{x^4}^2\right)^4}\,dx^1\,dx^1+\frac{x^2\,x^1}{\left({x^1}^2+{x^2}^2+{x^3}^2-{x^4}^2\right)^4}\,dx^1\,dx^2$$

$$+\frac{x^1\,x^3}{\left({x^1}^2+{x^2}^2+{x^3}^2-{x^4}^2\right)^4}\,dx^1\,dx^3-\frac{x^1\,x^4}{\left({x^1}^2+{x^2}^2+{x^3}^2-{x^4}^2\right)^4}\,dx^1\,dx^4$$

$$+\frac{{x^2}^2}{\left({x^1}^2+{x^2}^2+{x^3}^2-{x^4}^2\right)^4}\,dx^2\,dx^2+\frac{x^2\,x^3}{\left({x^1}^2+{x^2}^2+{x^3}^2-{x^4}^2\right)^4}\,dx^2\,dx^3$$

$$-\frac{x^2\,x^4}{\left({x^1}^2+{x^2}^2+{x^3}^2-{x^4}^2\right)^4}\,dx^2\,dx^4+\frac{{x^3}^2}{\left({x^1}^2+{x^2}^2+{x^3}^2-{x^4}^2\right)^4}\,dx^3\,dx^3$$

$$-\frac{x^3\,x^4}{\left({x^1}^2+{x^2}^2+{x^3}^2-{x^4}^2\right)^4}\,dx^3\,dx^4+\frac{{x^4}^2}{\left({x^1}^2+{x^2}^2+{x^3}^2-{x^4}^2\right)^4}\,dx^4\,dx^4$$

8. Determinants :

$\det(g)=-\frac{1}{8}\frac{{s_1}^3 s_2}{\left({x^1}^2+{x^2}^2+{x^3}^2-{x^4}^2\right)^6}$

9. Normalizers:

$\Phi_1=[x^1=x^1\xi_1, x^2=x^2\xi_1, x^3=x^3\xi_1, x^4=\xi_1 x^4]$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1,e],[e^2=1]]$

11. Petrov Reference: missing from Petrov

[7, 4, 4]

1. Reference : so(2,1)+so(2,1)+R

2. Lie Algebras (Isotropy Adapted, Structure Theory Adapted):

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7
Y_1	.	$-Y_6$	.	$-Y_7$	.	$-Y_2$	$-Y_4$
Y_2		.	.	$-Y_5$	$-Y_4$	Y_1	.
Y_3			.	.	.	.	.
Y_4				.	$-Y_2$	.	$-Y_1$
Y_5					.	Y_7	Y_6
Y_6						.	Y_5
Y_7							.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	.	e_3	$-e_2$	.	.	.	.
e_2		.	$-e_1$	.	.	.	.
e_3			.	.	.	.	.
e_4				.	e_6	$-e_5$	.
e_5					.	$-e_4$	.
e_6						.	.
e_7							.

3. Isomorphisms:

$$X_1 \to \frac{1}{2}\,Y_1 + \frac{1}{2}\,Y_5, \qquad X_5 \to \frac{1}{2}\,Y_2 + \frac{1}{2}\,Y_4 + \frac{1}{2}\,Y_6 + \frac{1}{2}\,Y_7,$$

$$X_2 \to -\frac{1}{2}\,Y_2 - \frac{1}{2}\,Y_4 + \frac{1}{2}\,Y_6 + \frac{1}{2}\,Y_7, \qquad X_6 \to -\frac{1}{4}\,Y_2 + \frac{1}{4}\,Y_4 + \frac{1}{4}\,Y_6 - \frac{1}{4}\,Y_7,$$

$$X_3 \to \frac{1}{4}\,Y_2 - \frac{1}{4}\,Y_4 + \frac{1}{4}\,Y_6 - \frac{1}{4}\,Y_7, \qquad X_7 \to Y_3.$$

$$X_4 \to -\frac{1}{2}\,Y_1 + \frac{1}{2}\,Y_5,$$

$$X_1 \to e_2, \qquad X_5 \to \frac{1}{2}\,e_4 - \frac{1}{2}\,e_6,$$

$$X_2 \to \frac{1}{2}\,e_1 - \frac{1}{2}\,e_3, \qquad X_6 \to e_4 + e_6,$$

$$X_3 \to e_1 + e_3, \qquad X_7 \to e_7.$$

$$X_4 \to e_5,$$

4. Isotropy: F4 $[-e_1 + e_4,\, e_3 + e_6,\, e_2 - e_5]$

5. Vector Fields Γ:

$$X_1 = -x^2\,\partial_{x^2} - \partial_{x^3} \qquad X_5 = -\frac{1}{2}\,\left(x^1\right)^2\,\partial_{x^1} - \frac{1}{2}\,\mathrm{e}^{x^3}\,\partial_{x^2} - x^1\,\partial_{x^3}$$

$$X_2 = \frac{1}{2}\,\partial_{x^2} \qquad X_6 = -\,\partial_{x^1}$$

$$X_3 = \mathrm{e}^{x^3}\,\partial_{x^1} + x^4\,\partial_{x^2} + 2\,x^2\,\partial_{x^3} \qquad X_7 = \partial_{x^4}$$

$$X_4 = x^1\,\partial_{x^1} + \partial_{x^3}$$

6. Base Point: $[0, 0, 0, 0]$

7. Γ Invariant Quadratic Forms: $g = s_i\sigma^i$

$$\sigma^1 = -2\,\mathrm{e}^{-x^3}\,dx^1\,dx^2 + dx^3\,dx^3 \qquad \sigma^2 = dx^4\,dx^4$$

8. Determinants :

$\det(g) = -4\,{s_1}^3\mathrm{e}^{-2\,x^3}\,s_2$

9. Normalizers:

$$\Phi_1 = [x^1 = x^1, x^2 = x^2, x^3 = x^3, x^4 = \xi_1 x^4]$$

10. Normalized Metrics (with respect to the invariant quadratic forms):

$[[s_1, e], [e^2 = 1]]$

11. Petrov Reference: missing from Petrov

A.2.8.3 $F6$

See Section 3.3.4 for information regarding this case.

A.3 Maple worksheet for G_6 on V_3

A.3.1 $F3$

Maple Worksheet
Six-dimensional Lie algebra
Three-dimensional Isotropy
Isotropy Type F3

These Maple worksheets aim to validate the claims of chapter 3 regarding the Schmidt method.

F3

We load the data structure for so(3) and initialize:

```
> LDs := SimpleLieAlgebraData("so(3)", algs, labelformat = "gl",
  labels = ['E', 'theta'], version = 2);
```

$$LDs := [e1, e2] = e3,\ [e1, e3] = -e2,\ [e2, e3] = e1,\ [E12, E13, E23],\ [\theta12, \theta13, \theta23] \tag{1.1}$$

```
> DGsetup(LDs):
```

We give the standard representation for so(3):

```
> so3 := StandardRepresentation(algs);
```

$$so3 := \left[\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\right] \tag{1.2}$$

Observe the brackets of the matrices in the standard representation,
which of course match exactly the structure equations
above:

```
> so3[1].so3[2] - so3[2].so3[1]; # [e4, e5] = e6
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{1.3}$$

```
> so3[1].so3[3] - so3[3].so3[1]; # [e4, e6] = -e5
```

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \tag{1.4}$$

```
> so3[2].so3[3] - so3[3].so3[2]; # [e5, e6] = e4
```

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.5}$$

The matrices of the representation of so(3) above
give the adjoint representation of the isotropy subalgebra
restricted to a reductive complement.
Also, if e4, e5, e6 represent the isotropy subalgebra, then we have

[e4, e5] = e6, [e4, e6] = -e5, [e5, e6] = e4 as the bracket relations (those of so(3)). Using this information we define a Lie algebra. Note that not all the brackets are determined at this point. By design, e1, e2, e3 form a reductive complement:

```
> LDx := LieAlgebraData([
  '[e1,e2]=a1*e1+a2*e2+a3*e3+a4*e4+a5*e5+a6*e6',
  '[e1,e3]=b1*e1+b2*e2+b3*e3+b4*e4+b5*e5+b6*e6',
  '[e1,e4]= -e2',
  '[e1,e5]= -e3',
  '[e1,e6]= 0',

  '[e2,e3]=c1*e1+c2*e2+c3*e3+c4*e4+c5*e5+c6*e6',
  '[e2,e4]= e1',
  '[e2,e5]= 0',
  '[e2,e6]= -e3',

  '[e3,e4]= 0',
  '[e3,e5]= e1',
  '[e3,e6]= e2',

  '[e4,e5]= e6',
  '[e4,e6]= -e5',

  '[e5,e6]= e4'],
  ['e1','e2','e3','e4','e5','e6'],algx);
```

$$LDx := [e1, e2] = a1\ e1 + a2\ e2 + a3\ e3 + a4\ e4 + a5\ e5 + a6\ e6,\ [e1, e3] = b1\ e1 + b2\ e2 + b3\ e3 + b4\ e4 + b5\ e5 + b6\ e6,\ [e1, e4] = -e2,\ [e1, e5] = -e3,\ [e1, e6] = 0,\ [e2, e3] = c1\ e1 + c2\ e2 + c3\ e3 + c4\ e4 + c5\ e5 + c6\ e6,\ [e2, e4] = e1,\ [e2, e5] = 0,\ [e2, e6] = -e3,\ [e3, e4] = 0,\ [e3, e5] = e1,\ [e3, e6] = e2,\ [e4, e5] = e6,\ [e4, e6] = -e5,\ [e5, e6] = e4 \tag{1.6}$$

Initialize the Lie algebra:

```
> DGsetup(LDx, [e], [theta]):
```

Note the adjoint representations of e4, e5, and e6 restricted to the reductive complement:

```
> Adjoint(e4, [e1,e2,e3]), Adjoint(e5, [e1,e2,e3]), Adjoint(e6,
  [e1,e2,e3]);
```

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{1.7}$$

We wish to solve for the unknowns in the bracket relations of the Lie algebra have created. This next set of routines will enforce the Jacobi identities and extract any linear equations in the unknowns:

```
> ChangeFrame(algx):
  lineqs := []:
  for i from 1 to 6 do
  lineqs||i := []:
   cfs := convert(DGinformation(
  ExteriorDerivative(ExteriorDerivative(theta||i)),
```

```
    "CoefficientSet"), list):
    tf := map(type, cfs, linear):
    for k from 1 to nops(cfs) do
     if tf[k] then
      lineqs||i := [op(lineqs||i), cfs[k]]:
     fi:
    od:
  od:
  for j from 1 to 6 do
    lineqs := [op(lineqs), op(lineqs||j)]:
  od:
  lineqs := convert(lineqs, set):
```

Here are the linear equations we wish to solve first:

```
> lineqs;
```

$$\{a2, a5, a6, b1, b3, b4, b6, c3, c4, c5, -a1, -a5, -a6, -b1, -b3, -b4, -b6, -c2, -c4, -c5, -a1 - c3, a1 + c3, -a2 + b3, a3 + b2, a3 - c1, -a4 + b5, a4 - c6, -a5 - b4, a6 + c4, -b1 + c2, -b2 - a3, b2 + c1, b4 + a5, -b5 + c6, -b6 - c5, -c1 - b2, -c4 - a6, c5 + b6\} \tag{1.8}$$

Solving:

```
> sol := solve(lineqs);
```

$$sol := \{a1 = 0, a2 = 0, a3 = -b2, a4 = c6, a5 = 0, a6 = 0, b1 = 0, b2 = b2, b3 = 0, b4 = 0, b5 = c6, b6 = 0, c1 = -b2, c2 = 0, c3 = 0, c4 = 0, c5 = 0, c6 = c6\} \tag{1.9}$$

We take the solution and substitute it into the original
Lie algebra structure equations:

```
> LDxx := eval(LDx, sol union {algx=algxx});
```

$$LDxx := [e1, e2] = -b2\, e3 + c6\, e4, [e1, e3] = b2\, e2 + c6\, e5, [e1, e4] = -e2, [e1, e5] = -e3, [e1, e6] = 0, [e2, e3] = -b2\, e1 + c6\, e6, [e2, e4] = e1, [e2, e5] = 0, [e2, e6] = -e3, [e3, e4] = 0, [e3, e5] = e1, [e3, e6] = e2, [e4, e5] = e6, [e4, e6] = -e5, [e5, e6] = e4 \tag{1.10}$$

```
> DGsetup(LDxx):
```

The remaining unknowns are then the following:

```
> par := indets(LDxx) minus {LDxx, algxx};
```

$$par := \{b2, c6\} \tag{1.11}$$

Here is the multiplication table for the Lie algebra:

```
> MultiplicationTable(algxx, "LieTable");
```

(1.12)

algxx	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$-b2\, e3 + c6\, e4$	$b2\, e2 + c6\, e5$	$-e2$	$-e3$	0
$e2$	$b2\, e3 - c6\, e4$	0	$-b2\, e1 + c6\, e6$	$e1$	0	$-e3$
$e3$	$-b2\, e2 - c6\, e5$	$b2\, e1 - c6\, e6$	0	0	$e1$	$e2$
$e4$	$e2$	$-e1$	0	0	$e6$	$-e5$
$e5$	$e3$	0	$-e1$	$-e6$	0	$e4$
$e6$	0	$e3$	$-e2$	$e5$	$-e4$	0

We enforce the Jacobi identities to extract any equations on the remaining unknowns:

```
> ddtheta1:=ExteriorDerivative(ExteriorDerivative(theta1));
  ddtheta2:=ExteriorDerivative(ExteriorDerivative(theta2));
  ddtheta3:=ExteriorDerivative(ExteriorDerivative(theta3));
  ddtheta4:=ExteriorDerivative(ExteriorDerivative(theta4));
  ddtheta5:=ExteriorDerivative(ExteriorDerivative(theta5));
  ddtheta6:=ExteriorDerivative(ExteriorDerivative(theta6));
```

$$ddtheta1 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta2 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta3 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta4 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta5 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta6 := 0\ \theta1 \wedge \theta2 \wedge \theta3 \qquad \textbf{(1.13)}$$

Note the Jacobi identities are satisfied. Make the following change of basis:

```
> LD1 := LieAlgebraData([e1+(1/2)*b2*e6, e2-(1/2)*b2*e5, e3+(1/2)*
  b2*e4,e4,e5,e6], alg1);
```

$$LD1 := \left[e1, e2\right] = \left(c6 + \frac{b2^2}{4}\right) e4,\ \left[e1, e3\right] = \left(c6 + \frac{b2^2}{4}\right) e5,\ [e1, e4] = -e2,\ [e1, e5] = -e3,\ [e1, e6] = 0,\ \left[e2, e3\right] = \left(c6 + \frac{b2^2}{4}\right) e6,\ [e2, e4] = e1,\ [e2, e5] = 0,\ [e2, e6] = -e3,\ [e3, e4] = 0,\ [e3, e5] = e1,\ [e3, e6] = e2,\ [e4, e5] = e6,\ [e4, e6] = -e5,\ [e5, e6] = e4 \qquad \textbf{(1.14)}$$

```
> DGsetup(LD1):
> MultiplicationTable(alg1, "LieTable");
```

alg1	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$\left(c6 + \frac{b2^2}{4}\right) e4$	$\left(c6 + \frac{b2^2}{4}\right) e5$	$-e2$	$-e3$	0
$e2$	$-\left(c6 + \frac{b2^2}{4}\right) e4$	0	$\left(c6 + \frac{b2^2}{4}\right) e6$	$e1$	0	$-e3$
$e3$	$-\left(c6 + \frac{b2^2}{4}\right) e5$	$-\left(c6 + \frac{b2^2}{4}\right) e6$	0	0	$e1$	$e2$
$e4$	$e2$	$-e1$	0	0	$e6$	$-e5$
$e5$	$e3$	0	$-e1$	$-e6$	0	$e4$
$e6$	0	$e3$	$-e2$	$e5$	$-e4$	0

(1.15)

We reparameterize and look for changes of basis that will help us classify the Lie algebra.

Let a = $c6 + \frac{b2^2}{4}$, and redefine the Lie algebra:

```
> LD2 := eval(LD1, {c6+(1/4)*b2^2 = a, alg1 = alg2});
```

$LD2 := [e1, e2] = a\, e4, [e1, e3] = a\, e5, [e1, e4] = -e2, [e1, e5] = -e3, [e1, e6] = 0, [e2, e3] = a\, e6, [e2, e4] = e1, [e2, e5] = 0, [e2, e6] = -e3, [e3, e4] = 0, [e3, e5] = e1, [e3, e6] = e2, [e4, e5] = e6, [e4, e6] = -e5, [e5, e6] = e4$ **(1.16)**

```
> DGsetup(LD2):
```

Observe the structure equations post reparameterization:

```
> MultiplicationTable(alg2, "LieTable");
```

alg2	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$a\, e4$	$a\, e5$	$-e2$	$-e3$	0
$e2$	$-a\, e4$	0	$a\, e6$	$e1$	0	$-e3$
$e3$	$-a\, e5$	$-a\, e6$	0	0	$e1$	$e2$
$e4$	$e2$	$-e1$	0	0	$e6$	$-e5$
$e5$	$e3$	0	$-e1$	$-e6$	0	$e4$
$e6$	0	$e3$	$-e2$	$e5$	$-e4$	0

(1.17)

Here is the Killing form the Lie algebra:

```
> kf := Killing(alg2);
```

$$kf := \begin{bmatrix} -4\,a & 0 & 0 & 0 & 0 & 0 \\ 0 & -4\,a & 0 & 0 & 0 & 0 \\ 0 & 0 & -4\,a & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \end{bmatrix} \quad \textbf{(1.18)}$$

The Killing form implies we need to investigate
three cases: a>0, a=0, and a<0.

Case 1: a>0.

Create a new Lie algebra name for this case:

```
> LD3 := eval(LD2, {alg2=alg3});
```

$LD3 := [e1, e2] = a\, e4, [e1, e3] = a\, e5, [e1, e4] = -e2, [e1, e5] = -e3, [e1, e6] = 0, [e2, e3] = a\, e6, [e2, e4] = e1, [e2, e5] = 0, [e2, e6] = -e3, [e3, e4] = 0, [e3, e5] = e1, [e3, e6] = e2, [e4, e5] = e6, [e4, e6] = -e5, [e5, e6] = e4$ **(1.19)**

```
> DGsetup(LD3):
```

Make the following change of basis,
which is valid under the assumption of
this case:

```
> LD3n := LieAlgebraData([e1/sqrt(a), e2/sqrt(a), e3/sqrt(a), e4,
  e5, e6], alg3n);
```

$LD3n := [e1, e2] = e4, [e1, e3] = e5, [e1, e4] = -e2, [e1, e5] = -e3, [e1, e6] = 0, [e2, e3$ **(1.20)**

$] = e6,\ [e2, e4] = e1,\ [e2, e5] = 0,\ [e2, e6] = -e3,\ [e3, e4] = 0,\ [e3, e5] = e1,\ [e3, e6$
$] = e2,\ [e4, e5] = e6,\ [e4, e6] = -e5,\ [e5, e6] = e4$

```
> DGsetup(LD3n):
```

Observe the structure equations contain no parameters
and the isotropy remains e4, e5, e6 with reductive
complement e1,e2,e3 in this basis:

```
> MultiplicationTable(alg3n, "LieTable");
```

alg3n	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$e4$	$e5$	$-e2$	$-e3$	0
$e2$	$-e4$	0	$e6$	$e1$	0	$-e3$
$e3$	$-e5$	$-e6$	0	0	$e1$	$e2$
$e4$	$e2$	$-e1$	0	0	$e6$	$-e5$
$e5$	$e3$	0	$-e1$	$-e6$	0	$e4$
$e6$	0	$e3$	$-e2$	$e5$	$-e4$	0

(1.21)

Now we wish to classify the Lie algebra.
Observe the Lie algebra is decomposable:

```
> Decompose(alg3n);
```

$$\left[\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}, [e1 - e6, e2 + e5, e3 - e4, e1 + e6, e2 - e5, e3\right. \tag{1.22}$$

$+ e4]$

Using the decomposition above,
we can easily find a change of basis
to put each factor in a standard form:

```
> ChangeFrame(alg3n):
> LD3nn := LieAlgebraData([1/2*(e1+e6), 1/2*(e2-e5), 1/2*(e3+e4),
  -1/2*(e1-e6), 1/2*(e2+e5), 1/2*(e3-e4)], alg3nn);
```

$$LD3nn := [e1, e2] = e3, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = 0, [e2, e6] = 0, [e3, e4] = 0, [e3, e5] = 0, [e3, e6] = 0, [e4, e5] = e6, [e4, e6] = -e5, [e5, e6] = e4 \tag{1.23}$$

```
> DGsetup(LD3nn):
```

Observe that the Lie algebra is the direct sum
so(3)⊕so(3):

```
> MultiplicationTable(alg3nn, "LieTable");
```

alg3nn	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$e3$	$-e2$	0	0	0
$e2$	$-e3$	0	$e1$	0	0	0
$e3$	$e2$	$-e1$	0	0	0	0
$e4$	0	0	0	0	$e6$	$-e5$
$e5$	0	0	0	$-e6$	0	$e4$
$e6$	0	0	0	$e5$	$-e4$	0

(1.24)

Next, we would like to verify that
the adh-invariant metric on g/h does not admit
any additional symmetries. To this end,
we change frame and set up the space
g/h (for g the Lie algebra and h the isotropy).

```
> ChangeFrame(alg3n):
> DGEnvironment[GSpace]([e1,e2,e3], [e4,e5,e6], G, vectorlabels =
  [X], formlabels = [sigma]);
```

(1.25)

$$G\ Space:\ G \tag{1.25}$$

Compute the symmetric tensors:

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3], 2);
```

$$S := \left[\sigma1 \otimes \sigma1, \frac{1}{2}\ \sigma1 \otimes \sigma2 + \frac{1}{2}\ \sigma2 \otimes \sigma1, \frac{1}{2}\ \sigma1 \otimes \sigma3 + \frac{1}{2}\ \sigma3 \otimes \sigma1, \sigma2 \otimes \sigma2, \frac{1}{2}\ \sigma2 \otimes \sigma3 + \frac{1}{2}\ \sigma3 \otimes \sigma2, \sigma3 \otimes \sigma3\right] \tag{1.26}$$

Calculate the metric preserved by h:

```
> g := InvariantGeometricObjectFields([X4, X5, X6], S);
```

$$g := \_C1\ \sigma1 \otimes \sigma1 + \_C1\ \sigma2 \otimes \sigma2 + \_C1\ \sigma3 \otimes \sigma3 \tag{1.27}$$

Next compute the isometry dimension of the metric:

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$6 \tag{1.28}$$

We conclude that the full isometry algebra of the metric
invariant under the action of the isotropy h, is
our six-dimensional Lie algebra.

Case 2: a=0.

Setup the Lie algebra of this case:

```
> LD4 := eval(LD2, {a=0, alg2=alg4});
```

$$LD4 := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = -e2, [e1, e5] = -e3, [e1, e6] = 0, [e2, e3] = 0, [e2, e4] = e1, [e2, e5] = 0, [e2, e6] = -e3, [e3, e4] = 0, [e3, e5] = e1, [e3, e6] = e2, [e4, e5] = e6, [e4, e6] = -e5, [e5, e6] = e4 \tag{1.29}$$

```
> DGsetup(LD4):
> MultiplicationTable(alg4, "LieTable");
```

alg4	e1	e2	e3	e4	e5	e6
e1	0	0	0	− e2	− e3	0
e2	0	0	0	e1	0	− e3
e3	0	0	0	0	e1	e2
e4	e2	− e1	0	0	e6	− e5
e5	e3	0	− e1	− e6	0	e4
e6	0	e3	− e2	e5	− e4	0

(1.30)

Observe the Levi-decomposition. The Lie algebra is the semidirect
sum of so(3) and the three-dimensional abelian Lie algebra:

```
> LeviDecomposition(alg4);
```

$$[[e1, e2, e3], [e4, e5, e6]] \tag{1.31}$$

We can check the isometry dimension first
if we wish:

```
> ChangeFrame(alg4):
> DGEnvironment[GSpace]([e1,e2,e3], [e4,e5,e6], G, vectorlabels =
  [X], formlabels = [sigma]);
```

$$G \text{ Space: } G \tag{1.32}$$

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3], 2);
```

$$S := \left[\sigma1 \otimes \sigma1, \frac{1}{2}\,\sigma1 \otimes \sigma2 + \frac{1}{2}\,\sigma2 \otimes \sigma1, \frac{1}{2}\,\sigma1 \otimes \sigma3 + \frac{1}{2}\,\sigma3 \otimes \sigma1, \sigma2 \otimes \sigma2, \frac{1}{2}\,\sigma2 \otimes \sigma3 + \frac{1}{2}\,\sigma3 \otimes \sigma2, \sigma3 \otimes \sigma3\right] \tag{1.33}$$

```
> g := InvariantGeometricObjectFields([X4, X5, X6], S);
```

$$g := \_C1\,\sigma1 \otimes \sigma1 + \_C1\,\sigma2 \otimes \sigma2 + \_C1\,\sigma3 \otimes \sigma3 \tag{1.34}$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$6 \tag{1.35}$$

Note we can show the Lie algebra is isomorphic to
the isometry algebra of the standard inner product on R^3.

Setup the standard metric of R^3:

```
> DGsetup([x,y,z], M):
> g := evalDG(dx&t dx + dy&t dy + dz&t dz);
```

$$g := dx \otimes dx + dy \otimes dy + dz \otimes dz \tag{1.36}$$

Compute the Killing vectors and Lie algebra:

```
> K := KillingVectors(g);
```

$$K := \left[-z\,\partial_x + x\,\partial_z, -z\,\partial_y + y\,\partial_z, \partial_z, -y\,\partial_x + x\,\partial_y, \partial_y, \partial_x\right] \tag{1.37}$$

There are six Killing vectors:

```
> nops(K);
```

$$6 \tag{1.38}$$

Compute and setup the structure equations:

```
> LDe := LieAlgebraData(K, euc);
```

$$LDe := [e1, e2] = -e4, [e1, e3] = e6, [e1, e4] = e2, [e1, e5] = 0, [e1, e6] = -e3, [e2, e3] = e5, [e2, e4] = -e1, [e2, e5] = -e3, [e2, e6] = 0, [e3, e4] = 0, [e3, e5] = 0, [e3, e6] = 0, [e4, e5] = e6, [e4, e6] = -e5, [e5, e6] = 0 \tag{1.39}$$

```
> DGsetup(LDe):
```

The following change of basis gives the exact
structure equations of LDe, proving the Lie algebra
is euc(3):

```
> ChangeFrame(alg4):
> LieAlgebraData([e1-e3-e5, -e2+e6, -e3, e2+e4, e2, -e1]);
```

$$[e1, e2] = -e4, [e1, e3] = e6, [e1, e4] = e2, [e1, e5] = 0, [e1, e6] = -e3, [e2, e3] = e5, [e2, e4] = -e1, [e2, e5] = -e3, [e2, e6] = 0, [e3, e4] = 0, [e3, e5] = 0, [e3, e6] = 0, [e4, e5] = e6, [e4, e6] = -e5, [e5, e6] = 0 \tag{1.40}$$

Case 3: a<0.

Setup the Lie algebra for this case:

```
> LD5 := eval(LD2, {alg2=alg5});
```

$$LD5 := [e1, e2] = a\,e4, [e1, e3] = a\,e5, [e1, e4] = -e2, [e1, e5] = -e3, [e1, e6] = 0, [e2, e3] = a\,e6, [e2, e4] = e1, [e2, e5] = 0, [e2, e6] = -e3, [e3, e4] = 0, [e3, e5] = e1, [e3, \tag{1.41}$$

$e6] = e2,\ [e4, e5] = e6,\ [e4, e6] = -e5,\ [e5, e6] = e4$

```
> DGsetup(LD5):
> MultiplicationTable(alg5, "LieTable");
```

alg5	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$a\,e4$	$a\,e5$	$-e2$	$-e3$	0
$e2$	$-a\,e4$	0	$a\,e6$	$e1$	0	$-e3$
$e3$	$-a\,e5$	$-a\,e6$	0	0	$e1$	$e2$
$e4$	$e2$	$-e1$	0	0	$e6$	$-e5$
$e5$	$e3$	0	$-e1$	$-e6$	0	$e4$
$e6$	0	$e3$	$-e2$	$e5$	$-e4$	0

(1.42)

```
> ChangeFrame(alg5):
```

This change of basis is real under the assumption a<0:

```
> LD5n := LieAlgebraData([1/sqrt(-a)*e1,1/sqrt(-a)*e2,1/sqrt(-a)*
  e3,e4,e5,e6], alg5n);
```

$$LD5n := [e1, e2] = -e4,\ [e1, e3] = -e5,\ [e1, e4] = -e2,\ [e1, e5] = -e3,\ [e1, e6] = 0,\ [e2, e3] = -e6,\ [e2, e4] = e1,\ [e2, e5] = 0,\ [e2, e6] = -e3,\ [e3, e4] = 0,\ [e3, e5] = e1,\ [e3, e6] = e2,\ [e4, e5] = e6,\ [e4, e6] = -e5,\ [e5, e6] = e4 \quad \textbf{(1.43)}$$

```
> DGsetup(LD5n):
```

Observe the structure equations contain no parameters:

```
> MultiplicationTable(alg5n, "LieTable");
```

alg5n	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$-e4$	$-e5$	$-e2$	$-e3$	0
$e2$	$e4$	0	$-e6$	$e1$	0	$-e3$
$e3$	$e5$	$e6$	0	0	$e1$	$e2$
$e4$	$e2$	$-e1$	0	0	$e6$	$-e5$
$e5$	$e3$	0	$-e1$	$-e6$	0	$e4$
$e6$	0	$e3$	$-e2$	$e5$	$-e4$	0

(1.44)

We check the isometry dimension of the adh-invariant metric on g/h:

```
> ChangeFrame(alg5n):
> DGEnvironment[GSpace]([e1,e2,e3], [e4,e5,e6], G, vectorlabels =
  [X], formlabels = [sigma]);
```

$$G\ Space{:}\ G \quad \textbf{(1.45)}$$

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3], 2);
```

$$S := \left[\sigma1 \otimes \sigma1,\ \frac{1}{2}\,\sigma1 \otimes \sigma2 + \frac{1}{2}\,\sigma2 \otimes \sigma1,\ \frac{1}{2}\,\sigma1 \otimes \sigma3 + \frac{1}{2}\,\sigma3 \otimes \sigma1,\ \sigma2 \otimes \sigma2,\ \frac{1}{2}\,\sigma2 \otimes \sigma3 + \frac{1}{2}\,\sigma3 \otimes \sigma2,\ \sigma3 \otimes \sigma3\right] \quad \textbf{(1.46)}$$

```
> g := InvariantGeometricObjectFields([X4, X5, X6], S);
```

$$g := \_C1\, \sigma 1 \otimes \sigma 1 + \_C1\, \sigma 2 \otimes \sigma 2 + \_C1\, \sigma 3 \otimes \sigma 3 \quad (1.47)$$

```
> IsometryAlgebraData(g, [], output = ["All"]);
```

$$6, [e1, e2] = -\_C1\, e4, [e1, e3] = -\_C1\, e5, \left[e1, e4\right] = -\frac{1}{\_C1}\, e2, \left[e1, e5\right] = -\frac{1}{\_C1}\, e3, [e1, e6] = 0, [e2, e3] = -\_C1\, e6, \left[e2, e4\right] = \frac{1}{\_C1}\, e1, [e2, e5] = 0, \left[e2, e6\right] = -\frac{1}{\_C1}\, e3, [e3, e4] = 0, \left[e3, e5\right] = \frac{1}{\_C1}\, e1, \left[e3, e6\right] = \frac{1}{\_C1}\, e2, \left[e4, e5\right] = \frac{1}{\_C1}\, e6, \left[e4, e6\right] = -\frac{1}{\_C1}\, e5, \left[e5, e6\right] = \frac{1}{\_C1}\, e4, [[0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1]] \quad (1.48)$$

The following change of basis shows our Lie algebra is so(3,1):

```
> ChangeFrame(alg5n):
> LieAlgebraData( [-e5, -e6, e4, -e3, -e1, -e2], alg5nx);
```

$$[e1, e2] = e3, [e1, e3] = -e2, [e1, e4] = e5, [e1, e5] = -e4, [e1, e6] = 0, [e2, e3] = e1, [e2, e4] = e6, [e2, e5] = 0, [e2, e6] = -e4, [e3, e4] = 0, [e3, e5] = e6, [e3, e6] = -e5, [e4, e5] = -e1, [e4, e6] = -e2, [e5, e6] = -e3 \quad (1.49)$$

```
> DGsetup(%);
```

Lie algebra: alg5nx **(1.50)**

```
> MultiplicationTable(alg5nx, "LieTable");
```

alg5nx	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$e3$	$-e2$	$e5$	$-e4$	0
$e2$	$-e3$	0	$e1$	$e6$	0	$-e4$
$e3$	$e2$	$-e1$	0	0	$e6$	$-e5$
$e4$	$-e5$	$-e6$	0	0	$-e1$	$-e2$
$e5$	$e4$	0	$-e6$	$e1$	0	$-e3$
$e6$	0	$e4$	$e5$	$e2$	$e3$	0

(1.51)

A.3.2 $F3$

Maple Worksheet
Six-dimensional Lie algebra
Three-dimensional Isotropy
Isotropy Type F4

These Maple worksheets aim to validate the claims of chapter 3 regarding the Schmidt method.

F4

Here we extract the structure equations for so(2,1) and initialize them:

```
> LDs := SimpleLieAlgebraData("so(2,1)", algs, labelformat = "gl",
  labels = ['E', 'theta'], version = 2);
```

$$LDs := [e1, e2] = e3,\ [e1, e3] = -e2,\ [e2, e3] = -e1,\ [E12, E13, E23],\ [\theta 12, \theta 13, \theta 23] \tag{1.1}$$

```
> DGsetup(LDs):
```

This next command gives the standard representation for so(2,1):

```
> so21 := StandardRepresentation(algs);
```

$$so21 := \left[\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\right] \tag{1.2}$$

Here are the bracket relations for the matrices, noting they of course match the structure equations of so(2,1):

```
> so21[1].so21[2] - so21[2].so21[1]; # [e4, e5] = e6
```

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{1.3}$$

```
> so21[1].so21[3] - so21[3].so21[1]; # [e4, e6] = -e5
```

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \tag{1.4}$$

```
> so21[2].so21[3] - so21[3].so21[2]; # [e5, e6] = -e4
```

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1.5}$$

The matrices of the representation of so(2,1) above give the adjoint representation of the isotropy subalgebra

restricted to a reductive complement.
Also, if e4, e5, e6 represent the isotropy subalgebra, then we have [e4, e5] = e6, [e4, e6] = -e5, [e5, e6] = -e4 as the bracket relations (those of so(2,1)). Using this information we define a Lie algebra. Note that not all the brackets are determined at this point. By design, e1, e2, e3 form a reductive complement:

```
alg2 > LDx := LieAlgebraData([
       '[e1,e2]=a1*e1+a2*e2+a3*e3+a4*e4+a5*e5+a6*e6',
       '[e1,e3]=b1*e1+b2*e2+b3*e3+b4*e4+b5*e5+b6*e6',
       '[e1,e4]= -e2',
       '[e1,e5]= -e3',
       '[e1,e6]= 0',

       '[e2,e3]=c1*e1+c2*e2+c3*e3+c4*e4+c5*e5+c6*e6',
       '[e2,e4]= e1',
       '[e2,e5]= 0',
       '[e2,e6]= -e3',

       '[e3,e4]= 0',
       '[e3,e5]= -e1',
       '[e3,e6]= -e2',
       '[e4,e5]= e6',
       '[e4,e6]= -e5',

       '[e5,e6]= -e4'],
       ['e1','e2','e3','e4','e5','e6'],algx);
```

$$LDx := [e1, e2] = a1\, e1 + a2\, e2 + a3\, e3 + a4\, e4 + a5\, e5 + a6\, e6,\ [e1, e3] = b1\, e1 + b2\, e2 + b3\, e3 + b4\, e4 + b5\, e5 + b6\, e6,\ [e1, e4] = -e2,\ [e1, e5] = -e3,\ [e1, e6] = 0,\ [e2, e3] = c1\, e1 + c2\, e2 + c3\, e3 + c4\, e4 + c5\, e5 + c6\, e6,\ [e2, e4] = e1,\ [e2, e5] = 0,\ [e2, e6] = -e3,\ [e3, e4] = 0,\ [e3, e5] = -e1,\ [e3, e6] = -e2,\ [e4, e5] = e6,\ [e4, e6] = -e5,\ [e5, e6] = -e4 \qquad \textbf{(1.6)}$$

Initialize the Lie algebra for use:

```
> DGsetup(LDx, [e], [theta]);
```

$$\text{Lie algebra: algx} \qquad \textbf{(1.7)}$$

Observe the adjoint representation of the isotropy restricted to the reductive complement e1,e2,e3 is as we designed:

```
> Adjoint(e4, [e1,e2,e3]), Adjoint(e5, [e1,e2,e3]), Adjoint(e6,
  [e1,e2,e3]);
```

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \textbf{(1.8)}$$

We wish to solve for the unknowns in the bracket relations of the Lie algebra have created. This next set of routines will enforce the Jacobi identities and extract any linear equations in the unknowns:

```
> ChangeFrame(algx):
```

```
lineqs := []:
for i from 1 to 6 do
lineqs||i := []:
 cfs := convert(DGinformation(
ExteriorDerivative(ExteriorDerivative(theta||i)),
 "CoefficientSet"), list):
 tf := map(type, cfs, linear):
 for k from 1 to nops(cfs) do
  if tf[k] then
   lineqs||i := [op(lineqs||i), cfs[k]]:
  fi:
 od:
od:
for j from 1 to 6 do
 lineqs := [op(lineqs), op(lineqs||j)]:
od:
lineqs := convert(lineqs, set):
```

Here are the linear equations:

```
> lineqs;
```

$$\{a1, a2, a6, b1, b4, b6, c3, c4, -a1, -a2, -a5, -b1, -b3, -b4, -c2, -c3, -c4, -c5, -a1 - c3, -a2 + b3, a2 - b3, -a3 + b2, -a3 - c1, -a4 + c6, a4 - b5, a5 - b4, -a6 + c4, -b1 + c2, -b2 + a3, b2 + c1, b4 - a5, -b5 + c6, b5 - a4, -b6 - c5, -c1 - b2, -c4 + a6, c5 + b6, -c6 + a4\} \tag{1.9}$$

Solve for the unknowns:

```
> sol := solve(lineqs);
```

$$sol := \{a1 = 0, a2 = 0, a3 = b2, a4 = a4, a5 = 0, a6 = 0, b1 = 0, b2 = b2, b3 = 0, b4 = 0, b5 = a4, b6 = 0, c1 = -b2, c2 = 0, c3 = 0, c4 = 0, c5 = 0, c6 = a4\} \tag{1.10}$$

Substitute the solution back into the Lie algebra,
giving it a new name and initialize:

```
> LDxx := eval(LDx, sol union {algx=algxx});
```

$$LDxx := [e1, e2] = b2\, e3 + a4\, e4,\ [e1, e3] = b2\, e2 + a4\, e5,\ [e1, e4] = -e2,\ [e1, e5] = -e3,\ [e1, e6] = 0,\ [e2, e3] = -b2\, e1 + a4\, e6,\ [e2, e4] = e1,\ [e2, e5] = 0,\ [e2, e6] = -e3,\ [e3, e4] = 0,\ [e3, e5] = -e1,\ [e3, e6] = -e2,\ [e4, e5] = e6,\ [e4, e6] = -e5,\ [e5, e6] = -e4 \tag{1.11}$$

```
> DGsetup(LDxx);
```

$$\textit{Lie algebra: algxx} \tag{1.12}$$

Here are the Jacobi identities noting
they are all satisfied at this point:

```
> ddtheta1:=ExteriorDerivative(ExteriorDerivative(theta1));
  ddtheta2:=ExteriorDerivative(ExteriorDerivative(theta2));
  ddtheta3:=ExteriorDerivative(ExteriorDerivative(theta3));
  ddtheta4:=ExteriorDerivative(ExteriorDerivative(theta4));
  ddtheta5:=ExteriorDerivative(ExteriorDerivative(theta5));
  ddtheta6:=ExteriorDerivative(ExteriorDerivative(theta6));
```

$$ddtheta1 := 0\, \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta2 := 0\, \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta3 := 0\, \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta4 := 0\ \theta 1 \wedge \theta 2 \wedge \theta 3$$

$$ddtheta5 := 0\ \theta 1 \wedge \theta 2 \wedge \theta 3$$

$$ddtheta6 := 0\ \theta 1 \wedge \theta 2 \wedge \theta 3 \qquad (1.13)$$

Here are the remaining unknowns in the structure equations:

```
> par := indets(LDxx) minus {LDxx, algxx};
```

$$par := \{a4, b2\} \qquad (1.14)$$

The structure equations are the following:

```
> MultiplicationTable(algxx, "LieTable");
```

algxx	e1	e2	e3	e4	e5	e6
e1	0	b2 e3 + a4 e4	b2 e2 + a4 e5	− e2	− e3	0
e2	− b2 e3 − a4 e4	0	− b2 e1 + a4 e6	e1	0	− e3
e3	− b2 e2 − a4 e5	b2 e1 − a4 e6	0	0	− e1	− e2
e4	e2	− e1	0	0	e6	− e5
e5	e3	0	e1	− e6	0	− e4
e6	0	e3	e2	e5	e4	0

(1.15)

Change basis:

```
> ChangeFrame(algxx):
> LD1 := LieAlgebraData([e1-(1/2)*b2*e6, e2+(1/2)*b2*e5, e3+(1/2)*
  b2*e4,e4,e5,e6], alg1);
```

$$LD1 := \left[e1, e2\right] = \left(a4 - \frac{b2^2}{4}\right) e4,\ \left[e1, e3\right] = \left(a4 - \frac{b2^2}{4}\right) e5,\ [e1, e4] = -e2,\ [e1, e5] = -e3,\ [e1, e6] = 0,\ \left[e2, e3\right] = \left(a4 - \frac{b2^2}{4}\right) e6,\ [e2, e4] = e1,\ [e2, e5] = 0,\ [e2, e6] = -e3,\ [e3, e4] = 0,\ [e3, e5] = -e1,\ [e3, e6] = -e2,\ [e4, e5] = e6,\ [e4, e6] = -e5,\ [e5, e6] = -e4 \qquad (1.16)$$

```
> DGsetup(LD1):
```

We reparameterize and look for further changes of basis that will help us classify the Lie algebra.

Let a = $a4 - \frac{b2^2}{4}$, and redefine the Lie algebra:

```
> LD2 := eval(LD1, {a4-(1/4)*b2^2 = a, alg1 = alg2});
```

$$LD2 := [e1, e2] = a\ e4,\ [e1, e3] = a\ e5,\ [e1, e4] = -e2,\ [e1, e5] = -e3,\ [e1, e6] = 0,\ [e2, e3] = a\ e6,\ [e2, e4] = e1,\ [e2, e5] = 0,\ [e2, e6] = -e3,\ [e3, e4] = 0,\ [e3, e5] = -e1,\ [e3, e6] = -e2,\ [e4, e5] = e6,\ [e4, e6] = -e5,\ [e5, e6] = -e4 \qquad (1.17)$$

```
> DGsetup(LD2):
> MultiplicationTable(alg2, "LieTable");
```

alg2	e1	e2	e3	e4	e5	e6
e1	0	a e4	a e5	− e2	− e3	0
e2	− a e4	0	a e6	e1	0	− e3
e3	− a e5	− a e6	0	0	− e1	− e2
e4	e2	− e1	0	0	e6	− e5
e5	e3	0	e1	− e6	0	− e4
e6	0	e3	e2	e5	e4	0

(1.18)

Observe the Killing form:

```
> kf := Killing(alg2);
```

$$kf := \begin{bmatrix} -4a & 0 & 0 & 0 & 0 & 0 \\ 0 & -4a & 0 & 0 & 0 & 0 \\ 0 & 0 & 4a & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \quad (1.19)$$

The Killing form implies we need to investigate three cases: a>0, a=0, and a<0.

Case 1: a<0.

```
> LD3 := eval(LD2, {alg2=alg3});
```

$LD3 := [e1, e2] = a\,e4,\ [e1, e3] = a\,e5,\ [e1, e4] = -e2,\ [e1, e5] = -e3,\ [e1, e6] = 0,\ [e2, e3] = a\,e6,\ [e2, e4] = e1,\ [e2, e5] = 0,\ [e2, e6] = -e3,\ [e3, e4] = 0,\ [e3, e5] = -e1,\ [e3, e6] = -e2,\ [e4, e5] = e6,\ [e4, e6] = -e5,\ [e5, e6] = -e4$ (1.20)

```
> DGsetup(LD3):
```

Observe the change of basis:

```
> LD3n := LieAlgebraData([1/sqrt(-a)*e1,1/sqrt(-a)*e2,1/sqrt(-a)*
  e3,e4,e5,e6], alg3n);
```

$LD3n := [e1, e2] = -e4,\ [e1, e3] = -e5,\ [e1, e4] = -e2,\ [e1, e5] = -e3,\ [e1, e6] = 0,\ [e2, e3] = -e6,\ [e2, e4] = e1,\ [e2, e5] = 0,\ [e2, e6] = -e3,\ [e3, e4] = 0,\ [e3, e5] = -e1,\ [e3, e6] = -e2,\ [e4, e5] = e6,\ [e4, e6] = -e5,\ [e5, e6] = -e4$ (1.21)

```
> DGsetup(LD3n):
> MultiplicationTable(alg3n, "LieTable");
```

(1.22)

alg3n	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$-e4$	$-e5$	$-e2$	$-e3$	0
$e2$	$e4$	0	$-e6$	$e1$	0	$-e3$
$e3$	$e5$	$e6$	0	0	$-e1$	$-e2$
$e4$	$e2$	$-e1$	0	0	$e6$	$-e5$
$e5$	$e3$	0	$e1$	$-e6$	0	$-e4$
$e6$	0	$e3$	$e2$	$e5$	$e4$	0

(1.22)

Observe the Lie algebra is decomposable:

```
> Query(alg3n, "Indecomposable");
```

$$false \quad \textbf{(1.23)}$$

We extract the structure of so(2,1) and then show our Lie algebra is the direct sum so(2,1)⊕so(2,1):

```
> LDs := SimpleLieAlgebraData("so(2,1)", so, version = 1);
```

$$LDs := [e1, e2] = e2, [e1, e3] = -e3, [e2, e3] = -e1 \quad \textbf{(1.24)}$$

```
> DGsetup(LDs);
```

$$\textit{Lie algebra: so} \quad \textbf{(1.25)}$$

The following change of basis shows our Lie algebra is indeed so(2,1)⊕so(2,1):

```
> LieAlgebraData( [(1/2)*e1-(1/2)*e6, (1/2)*e2-(1/2)*e3-(1/2)*e4+
  (1/2)*e5, -(1/4)*e2-(1/4)*e3-(1/4)*e4-(1/4)*e5, (1/2)*e1+(1/2)*
  e6, (1/2)*e2+(1/2)*e3-(1/2)*e4-(1/2)*e5, -(1/4)*e2+(1/4)*e3-(1/4)
  *e4+(1/4)*e5], alg3nx);
```

$$[e1, e2] = e2, [e1, e3] = -e3, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = 0, [e2, e3] = -e1, [e2, e4] = 0, [e2, e5] = 0, [e2, e6] = 0, [e3, e4] = 0, [e3, e5] = 0, [e3, e6] = 0, [e4, e5] = e5, [e4, e6] = -e6, [e5, e6] = -e4 \quad \textbf{(1.26)}$$

```
> DGsetup(%);
```

$$\textit{Lie algebra: alg3nx} \quad \textbf{(1.27)}$$

```
> MultiplicationTable(alg3nx, "LieTable");
```

alg3nx	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$e2$	$-e3$	0	0	0
$e2$	$-e2$	0	$-e1$	0	0	0
$e3$	$e3$	$e1$	0	0	0	0
$e4$	0	0	0	0	$e5$	$-e6$
$e5$	0	0	0	$-e5$	0	$-e4$
$e6$	0	0	0	$e6$	$e4$	0

(1.28)

We check the isometry dimension of the adh invariant metric on g/h:

```
> ChangeFrame(alg3n):
```

```
> S := GenerateSymmetricTensors([theta1, theta2, theta3],2)
```

$$S := \left[\theta 1 \otimes \theta 1, \frac{1}{2}\,\theta 1 \otimes \theta 2 + \frac{1}{2}\,\theta 2 \otimes \theta 1, \frac{1}{2}\,\theta 1 \otimes \theta 3 + \frac{1}{2}\,\theta 3 \otimes \theta 1, \theta 2 \otimes \theta 2, \frac{1}{2}\,\theta 2 \otimes \theta 3 + \frac{1}{2}\,\theta 3 \otimes \theta 2, \theta 3 \otimes \theta 3\right] \quad \textbf{(1.29)}$$

```
> DGEnvironment[GSpace]([e1,e2,e3], [e4,e5,e6], G, vectorlabels =
  [X], formlabels = [sigma]);
```

$$G\ Space\!:\ G \quad \textbf{(1.30)}$$

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3], 2);
```

$$S := \left[\sigma 1 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 2 + \frac{1}{2}\,\sigma 2 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 1, \sigma 2 \otimes \sigma 2, \frac{1}{2}\,\sigma 2 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 2, \sigma 3 \otimes \sigma 3\right] \quad \textbf{(1.31)}$$

```
> g := InvariantGeometricObjectFields([X4, X5, X6], S);
```

$$g := \_C1\, \sigma 1 \otimes \sigma 1 + \_C1\, \sigma 2 \otimes \sigma 2 - \_C1\, \sigma 3 \otimes \sigma 3 \quad \textbf{(1.32)}$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$6 \quad \textbf{(1.33)}$$

The isometry dimension is indeed 6.

Case 2: a=0.

Initialize the Lie algebra of this case:

```
> LD4 := eval(LD2, {a=0, alg2=alg4});
```

$$LD4 := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = -e2, [e1, e5] = -e3, [e1, e6] = 0, [e2, e3] = 0, [e2, e4] = e1, [e2, e5] = 0, [e2, e6] = -e3, [e3, e4] = 0, [e3, e5] = -e1, [e3, e6] = -e2, [e4, e5] = e6, [e4, e6] = -e5, [e5, e6] = -e4 \quad \textbf{(1.34)}$$

```
> DGsetup(LD4):
```

Note the Lie algebra admits a proper
Levi decomposition. The radical is always
returned first in the output:

```
> LeviDecomposition(alg4);
```

$$[[e1, e2, e3], [e4, e5, e6]] \quad \textbf{(1.35)}$$

Observe that the radical is three-dimensional abelian:

```
alg5 > ChangeFrame(alg4):
alg5 > LieAlgebraData([e1, e2, e3]);
```

$$[e1, e2] = 0, [e1, e3] = 0, [e2, e3] = 0 \quad \textbf{(1.36)}$$

```
alg5n > MultiplicationTable(alg4, "LieTable");
```

(1.37)

alg4	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	0	0	$-e2$	$-e3$	0
$e2$	0	0	0	$e1$	0	$-e3$
$e3$	0	0	0	0	$-e1$	$-e2$
$e4$	$e2$	$-e1$	0	0	$e6$	$-e5$
$e5$	$e3$	0	$e1$	$-e6$	0	$-e4$
$e6$	0	$e3$	$e2$	$e5$	$e4$	0

(1.37)

Therefore this Lie algebra is the semidirect sum
of the three-dimensional abelian Lie algebra and so(2,1).
That is, the Lie algebra is euc(2,1). We now check the
isometry dimension:

```
> DGEnvironment[GSpace]([e1,e2,e3], [e4,e5,e6], G, vectorlabels =
  [X], formlabels = [sigma]);
```

$$G\ Space\colon G \qquad \textbf{(1.38)}$$

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3], 2);
```

$$S := \left[\sigma1 \otimes \sigma1, \frac{1}{2}\,\sigma1 \otimes \sigma2 + \frac{1}{2}\,\sigma2 \otimes \sigma1, \frac{1}{2}\,\sigma1 \otimes \sigma3 + \frac{1}{2}\,\sigma3 \otimes \sigma1, \sigma2 \otimes \sigma2, \frac{1}{2}\,\sigma2 \otimes \sigma3 + \frac{1}{2}\,\sigma3 \otimes \sigma2, \sigma3 \otimes \sigma3\right] \qquad \textbf{(1.39)}$$

```
> g := InvariantGeometricObjectFields([X4, X5, X6], S);
```

$$g := \_C1\,\sigma1 \otimes \sigma1 + \_C1\,\sigma2 \otimes \sigma2 - \_C1\,\sigma3 \otimes \sigma3 \qquad \textbf{(1.40)}$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$6 \qquad \textbf{(1.41)}$$

Thus our Lie algebra is the full isometry algebra.

Case 3: a>0.

Initialize the Lie algebra of this case:

```
> LD5 := eval(LD2, {alg2=alg5});
```

$$LD5 := [e1, e2] = a\,e4,\ [e1, e3] = a\,e5,\ [e1, e4] = -e2,\ [e1, e5] = -e3,\ [e1, e6] = 0,\ [e2, e3] = a\,e6,\ [e2, e4] = e1,\ [e2, e5] = 0,\ [e2, e6] = -e3,\ [e3, e4] = 0,\ [e3, e5] = -e1,\ [e3, e6] = -e2,\ [e4, e5] = e6,\ [e4, e6] = -e5,\ [e5, e6] = -e4 \qquad \textbf{(1.42)}$$

```
> DGsetup(LD5):
> MultiplicationTable(alg5, "LieTable");
```

(1.43)

alg5	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$a\,e4$	$a\,e5$	$-e2$	$-e3$	0
$e2$	$-a\,e4$	0	$a\,e6$	$e1$	0	$-e3$
$e3$	$-a\,e5$	$-a\,e6$	0	0	$-e1$	$-e2$
$e4$	$e2$	$-e1$	0	0	$e6$	$-e5$
$e5$	$e3$	0	$e1$	$-e6$	0	$-e4$
$e6$	0	$e3$	$e2$	$e5$	$e4$	0

(1.43)

Make the following change of basis
and observe the structure equations
contain no parameter:

```
> LD5n := LieAlgebraData([1/sqrt(a)*e1,1/sqrt(a)*e2,1/sqrt(a)*e3,
  e4,e5,e6], alg5n);
```

$LD5n := [e1, e2] = e4, [e1, e3] = e5, [e1, e4] = -e2, [e1, e5] = -e3, [e1, e6] = 0, [e2, e3] = e6, [e2, e4] = e1, [e2, e5] = 0, [e2, e6] = -e3, [e3, e4] = 0, [e3, e5] = -e1, [e3, e6] = -e2, [e4, e5] = e6, [e4, e6] = -e5, [e5, e6] = -e4$ **(1.44)**

```
> DGsetup(LD5n):
> MultiplicationTable(alg5n, "LieTable");
```

alg5n	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$e4$	$e5$	$-e2$	$-e3$	0
$e2$	$-e4$	0	$e6$	$e1$	0	$-e3$
$e3$	$-e5$	$-e6$	0	0	$-e1$	$-e2$
$e4$	$e2$	$-e1$	0	0	$e6$	$-e5$
$e5$	$e3$	0	$e1$	$-e6$	0	$-e4$
$e6$	0	$e3$	$e2$	$e5$	$e4$	0

(1.45)

The following change of basis shows the Lie algebra
is so(3,1):

```
> LieAlgebraData( [e2,  e1, -e4, e3, e6, e5], alg5nx);
```

$[e1, e2] = e3, [e1, e3] = -e2, [e1, e4] = e5, [e1, e5] = -e4, [e1, e6] = 0, [e2, e3] = e1, [e2, e4] = e6, [e2, e5] = 0, [e2, e6] = -e4, [e3, e4] = 0, [e3, e5] = e6, [e3, e6] = -e5, [e4, e5] = -e1, [e4, e6] = -e2, [e5, e6] = -e3$ **(1.46)**

```
so > DGsetup(%);
```

Lie algebra: alg5nx **(1.47)**

```
> MultiplicationTable(alg5nx, "LieTable");
```

(1.48)

alg5nx	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$e3$	$-e2$	$e5$	$-e4$	0
$e2$	$-e3$	0	$e1$	$e6$	0	$-e4$
$e3$	$e2$	$-e1$	0	0	$e6$	$-e5$
$e4$	$-e5$	$-e6$	0	0	$-e1$	$-e2$
$e5$	$e4$	0	$-e6$	$e1$	0	$-e3$
$e6$	0	$e4$	$e5$	$e2$	$e3$	0

(1.48)

We check the isometry dimension of the adh-invariant metric on g/h:

```
> ChangeFrame(alg5n):
> DGEnvironment[GSpace]([e1,e2,e3], [e4,e5,e6], G, vectorlabels =
  [X], formlabels = [sigma]);
```

G Space: G **(1.49)**

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3], 2);
```

$$S := \left[\sigma1 \otimes \sigma1, \frac{1}{2}\,\sigma1 \otimes \sigma2 + \frac{1}{2}\,\sigma2 \otimes \sigma1, \frac{1}{2}\,\sigma1 \otimes \sigma3 + \frac{1}{2}\,\sigma3 \otimes \sigma1, \sigma2 \otimes \sigma2, \frac{1}{2}\,\sigma2 \otimes \sigma3 + \frac{1}{2}\,\sigma3 \otimes \sigma2, \sigma3 \otimes \sigma3\right] \quad \textbf{(1.50)}$$

```
> g := InvariantGeometricObjectFields([X4, X5, X6], S);
```

$$g := \_C1\,\sigma1 \otimes \sigma1 + \_C1\,\sigma2 \otimes \sigma2 - \_C1\,\sigma3 \otimes \sigma3 \quad \textbf{(1.51)}$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

6 **(1.52)**

A.4 Maple worksheet for G_6 on V_4

A.4.1 $F8$

Maple Worksheet
Six-dimensional Lie algebra
Two-dimensional Isotropy
Isotropy Type F8

These Maple worksheets aim to validate the claims of chapter 3 regarding the Schmidt method.

Here is a basis of so(3,1):

```
> B1,B2,B3,B4,B5,B6:=op(IsotropyType(output="SO31II"));
```

$$B1, B2, B3, B4, B5, B6 := \begin{bmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \textbf{(1)}$$

The following are the two-dimensional subalgebras of so(3,1):

F8: {B2 ,B3}
F9: {B1, B2}
F10: {B3, B4}

F8

A basis for the subalgebra F8 of so(3,1) is the following:

```
> B2a := 1/2*B2:
> [B2a ,B3];
```

$$\left[\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}\right] \quad \textbf{(1.1)}$$

Observe the bracket relation:

```
> B2a.B3-B3.B2a; # = e6
```

$$\begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(1.2)}$$

By assuming the above matrices define the adjoint action of e5 and e6 respectively, restricted to a reductive complement m = span({e1, e2, e3, e4}), we obtain the following structure equations:

```
> LDx := LieAlgebraData([
  '[e1,e2]=a1*e1+a2*e2+a3*e3+a4*e4+a5*e5+a6*e6',
  '[e1,e3]=b1*e1+b2*e2+b3*e3+b4*e4+b5*e5+b6*e6',
  '[e1,e4]=d1*e1+d2*e2+d3*e3+d4*e4+d5*e5+d6*e6',
  '[e1,e5]= 0',
  '[e1,e6]= -e3+e4',

  '[e2,e3]=c1*e1+c2*e2+c3*e3+c4*e4+c5*e5+c6*e6',
  '[e2,e4]=g1*e1+g2*e2+g3*e3+g4*e4+g5*e5+g6*e6',
  '[e2,e5]= 0',
  '[e2,e6]= 0',

  '[e3,e4]=h1*e1+h2*e2+h3*e3+h4*e4+h5*e5+h6*e6',
  '[e3,e5]= e4',
  '[e3,e6]= e1',

  '[e4,e5]= e3',
  '[e4,e6]= e1',

  '[e5,e6]= e6'],
  ['e1','e2','e3','e4','e5','e6'],algx);
```

$$LDx := [e1, e2] = a1\ e1 + a2\ e2 + a3\ e3 + a4\ e4 + a5\ e5 + a6\ e6,\ [e1, e3] = b1\ e1 + b2\ e2 + b3\ e3 + b4\ e4 + b5\ e5 + b6\ e6,\ [e1, e4] = d1\ e1 + d2\ e2 + d3\ e3 + d4\ e4 + d5\ e5 + d6\ e6,\ [e1, e5] = 0,\ [e1, e6] = -e3 + e4,\ [e2, e3] = c1\ e1 + c2\ e2 + c3\ e3 + c4\ e4 + c5\ e5 + c6\ e6,\ [e2, e4] = g1\ e1 + g2\ e2 + g3\ e3 + g4\ e4 + g5\ e5 + g6\ e6,\ [e2, e5] = 0,\ [e2, e6] = 0,\ [e3, e4] = h1\ e1 + h2\ e2 + h3\ e3 + h4\ e4 + h5\ e5 + h6\ e6,\ [e3, e5] = e4,\ [e3, e6] = e1,\ [e4, e5] = e3,\ [e4, e6] = e1,\ [e5, e6] = e6 \quad \textbf{(1.3)}$$

Initialize the Lie algebra:

```
> DGsetup(LDx, [e], [theta]);
```

Lie algebra: algx **(1.4)**

Observe the adjoint representations of the isotropy restricted to the reductive complement:

```
> Adjoint(e5, [e1,e2,e3,e4]), Adjoint(e6, [e1,e2,e3,e4]);
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(1.5)}$$

Extract the linear equations needing solving upon imposing the Jacobi identities:

```
> ChangeFrame(algx):
  lineqs := []:
  for i from 1 to 6 do
  lineqs||i := []:
```

```
 cfs := convert(DGinformation
(ExteriorDerivative(ExteriorDerivative(theta||i)),
 "CoefficientSet"), list):
 tf := map(type, cfs, linear):
 cnsts := map(type, cfs, constant):
 if nops(cnsts)>= 1 then
  for l from 1 to nops(cnsts) do
   if cnsts[l] then
    error cnsts[l], "Contradiction, there is a constant
coefficient.";
   fi;
  od;
 fi;
 for k from 1 to nops(cfs) do
  if tf[k] then
   lineqs||i := [op(lineqs||i), cfs[k]]:
  fi:
 od:
od:
for j from 1 to 6 do
 lineqs := [op(lineqs), op(lineqs||j)]:
od:
lineqs := convert(lineqs, set):
```

Here are the linear equations:

```
> lineqs;
```

$$\{a2, a3, a4, a5, h2, h3, h4, h5, -a6, -b1, -b2, -b5, -c1, -c2, -c5, -d1, -d2, -d5, -g1, -g2, -g5, -h6, a3 - c1, a4 + g1, a6 + c5, a6 + g5, -b1 + h3, b1 + h4, -b3 + d4, b3 - d4, b4 - d3, c1 + a4, -c2 + g2, -c3 + g4, c3 - g4, c4 - g3, -c5 + g5, -d1 + h3, d1 + h4, -d2 + b2, d3 - b4, -d5 + b5, -d6 - b6, -g1 + a3, g3 - c4, -g6 - c6, h6 + b5, h6 + d5, -a1 - c3 + g3, a1 + c3 + c4, a1 - c4 + g4, a1 + g3 + g4, b3 + b4 + h1, -c6 + g6 + a5, d3 + h1 + d4, -d6 + b6 + h5, -h1 - d3 + b3, h1 - d4 + b4, a3 - c1 + a4 + g1, -d1 + b1 + h3 + h4\} \tag{1.6}$$

We solve for the unknowns:

```
> sol := solve(lineqs);
```

$$sol := \{a1 = a1, a2 = 0, a3 = 0, a4 = 0, a5 = 0, a6 = 0, b1 = 0, b2 = 0, b3 = 0, b4 = -h1, b5 = 0, b6 = 0, c1 = 0, c2 = 0, c3 = -a1, c4 = 0, c5 = 0, c6 = 0, d1 = 0, d2 = 0, d3 = -h1, d4 = 0, d5 = 0, d6 = 0, g1 = 0, g2 = 0, g3 = 0, g4 = -a1, g5 = 0, g6 = 0, h1 = h1, h2 = 0, h3 = 0, h4 = 0, h5 = 0, h6 = 0\} \tag{1.7}$$

We substitute the solution into the Lie algebra structure equations and initialize:

```
> LDxx := eval(LDx, sol union {algx=algxx});
```

$$LDxx := [e1, e2] = a1\ e1,\ [e1, e3] = -h1\ e4,\ [e1, e4] = -h1\ e3,\ [e1, e5] = 0,\ [e1, e6] = -e3 + e4,\ [e2, e3] = -a1\ e3,\ [e2, e4] = -a1\ e4,\ [e2, e5] = 0,\ [e2, e6] = 0,\ [e3, e4] = h1\ e1,\ [e3, e5] = e4,\ [e3, e6] = e1,\ [e4, e5] = e3,\ [e4, e6] = e1,\ [e5, e6] = e6 \tag{1.8}$$

```
> DGsetup(LDxx):
```

Here are the remaining unknowns:

```
> par := indets(LDxx) minus {LDxx, algxx};
```

(1.9)

$$par := \{a1, h1\} \tag{1.9}$$

And the structure equations:

```
> MultiplicationTable(algxx, "LieTable");
```

algxx	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$a1\,e1$	$-h1\,e4$	$-h1\,e3$	0	$-e3+e4$
$e2$	$-a1\,e1$	0	$-a1\,e3$	$-a1\,e4$	0	0
$e3$	$h1\,e4$	$a1\,e3$	0	$h1\,e1$	$e4$	$e1$
$e4$	$h1\,e3$	$a1\,e4$	$-h1\,e1$	0	$e3$	$e1$
$e5$	0	0	$-e4$	$-e3$	0	$e6$
$e6$	$e3-e4$	0	$-e1$	$-e1$	$-e6$	0

(1.10)

We check the Jacobi identities for conditions
on the remaining unknowns:

```
> ddtheta1:=ExteriorDerivative(ExteriorDerivative(theta1));
  ddtheta2:=ExteriorDerivative(ExteriorDerivative(theta2));
  ddtheta3:=ExteriorDerivative(ExteriorDerivative(theta3));
  ddtheta4:=ExteriorDerivative(ExteriorDerivative(theta4));
  ddtheta5:=ExteriorDerivative(ExteriorDerivative(theta5));
  ddtheta6:=ExteriorDerivative(ExteriorDerivative(theta6));
```

$$ddtheta1 := -h1\,a1\,\theta2 \wedge \theta3 \wedge \theta4$$

$$ddtheta2 := 0\,\theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta3 := -h1\,a1\,\theta1 \wedge \theta2 \wedge \theta4$$

$$ddtheta4 := -h1\,a1\,\theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta5 := 0\,\theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta6 := 0\,\theta1 \wedge \theta2 \wedge \theta3 \tag{1.11}$$

We see we must have a1=0 or h1=0, therefore we case split.

Case 1: a1=0, h1=0
This case is gives a flat metric.

```
> ChangeFrame(algxx):
> LD1n := eval(LieAlgebraData([e1,e2,e3,e4,e5,e6], alg1n), {a1=0,
  h1=0});
```

$$LD1n := [e1, e2] = 0,\ [e1, e3] = 0,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e1, e6] = -e3 + e4,\ [e2, e3] = 0,\ [e2, e4] = 0,\ [e2, e5] = 0,\ [e2, e6] = 0,\ [e3, e4] = 0,\ [e3, e5] = e4,\ [e3, e6] = e1,\ [e4, e5] = e3,\ [e4, e6] = e1,\ [e5, e6] = e6 \tag{1.12}$$

```
> DGsetup(LD1n):
```

We initialize the homogeneous space:

```
> DGEnvironment[GSpace]([e1,e2,e3,e4], [e5,e6], G, vectorlabels =
  [X], formlabels = [sigma]);
```

$$G\ Space{:}\ G \tag{1.13}$$

We need the symmetric tensors to compute the general metric:

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4],
```

```
2);
```

$$S := \left[\sigma 1 \otimes \sigma 1, \frac{1}{2}\, \sigma 1 \otimes \sigma 2 + \frac{1}{2}\, \sigma 2 \otimes \sigma 1, \frac{1}{2}\, \sigma 1 \otimes \sigma 3 + \frac{1}{2}\, \sigma 3 \otimes \sigma 1, \frac{1}{2}\, \sigma 1 \otimes \sigma 4 + \frac{1}{2}\, \sigma 4 \otimes \sigma 1, \sigma 2 \otimes \sigma 2, \frac{1}{2}\, \sigma 2 \otimes \sigma 3 + \frac{1}{2}\, \sigma 3 \otimes \sigma 2, \frac{1}{2}\, \sigma 2 \otimes \sigma 4 + \frac{1}{2}\, \sigma 4 \otimes \sigma 2, \sigma 3 \otimes \sigma 3, \frac{1}{2}\, \sigma 3 \otimes \sigma 4 + \frac{1}{2}\, \sigma 4 \otimes \sigma 3, \sigma 4 \otimes \sigma 4\right] \tag{1.14}$$

Here is the general metric:

```
> g := InvariantGeometricObjectFields([X5, X6], S);
```

$$g := \_C1\, \sigma 1 \otimes \sigma 1 + \_C2\, \sigma 2 \otimes \sigma 2 + \_C1\, \sigma 3 \otimes \sigma 3 - \_C1\, \sigma 4 \otimes \sigma 4 \tag{1.15}$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$10 \tag{1.16}$$

The isometry dimension is 10, thus we check the curvature tensor.
Indeed it vanishes and therefore the metric is flat.

```
> CurvatureTensor(g);
```

$$0\, X1 \otimes \sigma 1 \otimes \sigma 1 \otimes \sigma 1 \tag{1.17}$$

Case 2: a1=0, h1<>0

```
> ChangeFrame(algxx):
> LD2 := eval(LieAlgebraData([e1,e2,e3,e4,e5,e6], alg2), {a1=0, h1=
  h1});
```

$$LD2 := [e1, e2] = 0, [e1, e3] = -h1\, e4, [e1, e4] = -h1\, e3, [e1, e5] = 0, [e1, e6] = -e3 + e4, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = 0, [e2, e6] = 0, [e3, e4] = h1\, e1, [e3, e5] = e4, [e3, e6] = e1, [e4, e5] = e3, [e4, e6] = e1, [e5, e6] = e6 \tag{1.18}$$

```
> DGsetup(LD2):
```

As seen just below, the invariant metric admits a seven dimensional isometry algebra.

```
> ChangeFrame(alg2):
> DGEnvironment[GSpace]([e1,e2,e3,e4], [e5,e6], G, vectorlabels =
  [X], formlabels = [sigma]);
```

$$G\ Space{:}\ G \tag{1.19}$$

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4],
  2);
```

$$S := \left[\sigma 1 \otimes \sigma 1, \frac{1}{2}\, \sigma 1 \otimes \sigma 2 + \frac{1}{2}\, \sigma 2 \otimes \sigma 1, \frac{1}{2}\, \sigma 1 \otimes \sigma 3 + \frac{1}{2}\, \sigma 3 \otimes \sigma 1, \frac{1}{2}\, \sigma 1 \otimes \sigma 4 + \frac{1}{2}\, \sigma 4 \otimes \sigma 1, \sigma 2 \otimes \sigma 2, \frac{1}{2}\, \sigma 2 \otimes \sigma 3 + \frac{1}{2}\, \sigma 3 \otimes \sigma 2, \frac{1}{2}\, \sigma 2 \otimes \sigma 4 + \frac{1}{2}\, \sigma 4 \otimes \sigma 2, \sigma 3 \otimes \sigma 3, \frac{1}{2}\, \sigma 3 \otimes \sigma 4 + \frac{1}{2}\, \sigma 4 \otimes \sigma 3, \sigma 4 \otimes \sigma 4\right] \tag{1.20}$$

```
> g := InvariantGeometricObjectFields([X5, X6], S);
```

$$g := \_C1\, \sigma 1 \otimes \sigma 1 + \_C2\, \sigma 2 \otimes \sigma 2 + \_C1\, \sigma 3 \otimes \sigma 3 - \_C1\, \sigma 4 \otimes \sigma 4 \tag{1.21}$$

```
> IsometryAlgebraData(g, [], output = [ "Dimension"], algx);
```

$$7 \tag{1.22}$$

Therefore this case is not relevant to our work and is excluded.

It can be shown that the seven dimensional isometry is that of
[7,4,3].

Case 3: a1<>0, h1=0
Admits 10-dimensional group of symmetries.

```
> ChangeFrame(algxx):
> LD3 := eval(LieAlgebraData([e1,e2,e3,e4,e5,e6], alg3), {a1=a1,
  h1=0});
```

$$LD3 := [e1, e2] = a1\ e1,\ [e1, e3] = 0,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e1, e6] = -e3 + e4,\ [e2, e3] = -a1\ e3,\ [e2, e4] = -a1\ e4,\ [e2, e5] = 0,\ [e2, e6] = 0,\ [e3, e4] = 0,\ [e3, e5] = e4,\ [e3, e6] = e1,\ [e4, e5] = e3,\ [e4, e6] = e1,\ [e5, e6] = e6 \quad (1.23)$$

```
> DGsetup(LD3):
```

Make a change of basis:

```
> LD3n := LieAlgebraData([e1,1/a1*e2,e3,e4,e5,e6], alg3n);
```

$$LD3n := [e1, e2] = e1,\ [e1, e3] = 0,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e1, e6] = -e3 + e4,\ [e2, e3] = -e3,\ [e2, e4] = -e4,\ [e2, e5] = 0,\ [e2, e6] = 0,\ [e3, e4] = 0,\ [e3, e5] = e4,\ [e3, e6] = e1,\ [e4, e5] = e3,\ [e4, e6] = e1,\ [e5, e6] = e6 \quad (1.24)$$

```
> DGsetup(LD3n):
```

We check the isometry dimension:

```
> DGEnvironment[GSpace]([e1,e2,e3,e4], [e5,e6], G, vectorlabels =
  [X], formlabels = [sigma]);
```

$$G\ Space:\ G \quad (1.25)$$

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4],
  2);
```

$$S := \left[\sigma1 \otimes \sigma1,\ \frac{1}{2}\,\sigma1 \otimes \sigma2 + \frac{1}{2}\,\sigma2 \otimes \sigma1,\ \frac{1}{2}\,\sigma1 \otimes \sigma3 + \frac{1}{2}\,\sigma3 \otimes \sigma1,\ \frac{1}{2}\,\sigma1 \otimes \sigma4 + \frac{1}{2}\,\sigma4 \otimes \sigma1,\ \sigma2 \otimes \sigma2,\ \frac{1}{2}\,\sigma2 \otimes \sigma3 + \frac{1}{2}\,\sigma3 \otimes \sigma2,\ \frac{1}{2}\,\sigma2 \otimes \sigma4 + \frac{1}{2}\,\sigma4 \otimes \sigma2,\ \sigma3 \otimes \sigma3,\ \frac{1}{2}\,\sigma3 \otimes \sigma4 + \frac{1}{2}\,\sigma4 \otimes \sigma3,\ \sigma4 \otimes \sigma4\right] \quad (1.26)$$

Here is the general metric:

```
> g := InvariantGeometricObjectFields([X5, X6], S);
```

$$g := \_C1\ \sigma1 \otimes \sigma1 + \_C2\ \sigma2 \otimes \sigma2 + \_C1\ \sigma3 \otimes \sigma3 - \_C1\ \sigma4 \otimes \sigma4 \quad (1.27)$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$10 \quad (1.28)$$

We next investigate the nature of the metric.
Note the curvature tensor is non-zero:

```
> C := CurvatureTensor(g);
```

$$C := -X1 \otimes \sigma2 \otimes \sigma1 \otimes \sigma2 + X1 \otimes \sigma2 \otimes \sigma2 \otimes \sigma1 - \frac{\_C1}{\_C2}\,X1 \otimes \sigma3 \otimes \sigma1 \otimes \sigma3 + \frac{\_C1}{\_C2}\,X1 \otimes \sigma3 \otimes \sigma3 \otimes \sigma1 + \frac{\_C1}{\_C2}\,X1 \otimes \sigma4 \otimes \sigma1 \otimes \sigma4 - \frac{\_C1}{\_C2}\,X1 \otimes \sigma4 \otimes \sigma4 \quad (1.29)$$

$$\otimes\,\sigma 1 + \frac{\_C1}{\_C2}\,X2 \otimes \sigma 1 \otimes \sigma 1 \otimes \sigma 2 - \frac{\_C1}{\_C2}\,X2 \otimes \sigma 1 \otimes \sigma 2 \otimes \sigma 1 - \frac{\_C1}{\_C2}\,X2 \otimes \sigma 3 \otimes \sigma 2 \otimes \sigma 3 + \frac{\_C1}{\_C2}\,X2 \otimes \sigma 3 \otimes \sigma 3 \otimes \sigma 2 + \frac{\_C1}{\_C2}\,X2 \otimes \sigma 4 \otimes \sigma 2 \otimes \sigma 4 - \frac{\_C1}{\_C2}\,X2 \otimes \sigma 4 \otimes \sigma 4 \otimes \sigma 2 + \frac{\_C1}{\_C2}\,X3 \otimes \sigma 1 \otimes \sigma 1 \otimes \sigma 3 - \frac{\_C1}{\_C2}\,X3 \otimes \sigma 1 \otimes \sigma 3 \otimes \sigma 1 + X3 \otimes \sigma 2 \otimes \sigma 2 \otimes \sigma 3 - X3 \otimes \sigma 2 \otimes \sigma 3 \otimes \sigma 2 + \frac{\_C1}{\_C2}\,X3 \otimes \sigma 4 \otimes \sigma 3 \otimes \sigma 4 - \frac{\_C1}{\_C2}\,X3 \otimes \sigma 4 \otimes \sigma 4 \otimes \sigma 3 + \frac{\_C1}{\_C2}\,X4 \otimes \sigma 1 \otimes \sigma 1 \otimes \sigma 4 - \frac{\_C1}{\_C2}\,X4 \otimes \sigma 1 \otimes \sigma 4 \otimes \sigma 1 + X4 \otimes \sigma 2 \otimes \sigma 2 \otimes \sigma 4 - X4 \otimes \sigma 2 \otimes \sigma 4 \otimes \sigma 2 + \frac{\_C1}{\_C2}\,X4 \otimes \sigma 3 \otimes \sigma 3 \otimes \sigma 4 - \frac{\_C1}{\_C2}\,X4 \otimes \sigma 3 \otimes \sigma 4 \otimes \sigma 3$$

However, it is covariantly constant:

```
> CovariantDerivative(C, Christoffel(g) );
```

$$0\,X1 \otimes \sigma 1 \otimes \sigma 1 \otimes \sigma 1 \otimes \sigma 1 \qquad \textbf{(1.30)}$$

Therefore the metric has constant curvature and 10-dimensional isometry algebra and we exclude this case.
This concludes the investigation into the isotropy type F8.

A.4.2 $F9$

Maple Worksheet
Six-dimensional Lie algebra
Two-dimensional Isotropy
Isotropy Type F9

These Maple worksheets aim to validate the claims of chapter 3 regarding the Schmidt method.

F9

A basis for the subalgebra F9 of so(3,1) is the following:

```
> B1a, B2a := 1/2*B1, 1/2*B2;
```

$$B1a, B2a := \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{1.1}$$

Note the subalgebra is abelian:

```
> B1a.B2a-B2a.B1a;
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.2}$$

By assuming the above matrices define the adjoint action of e5 and e6 respectively, restricted to a reductive complement m = span({e1, e2, e3, e4}), we obtain the following structure equations:

```
> LDx := LieAlgebraData([
  '[e1,e2]=a1*e1+a2*e2+a3*e3+a4*e4+a5*e5+a6*e6',
  '[e1,e3]=b1*e1+b2*e2+b3*e3+b4*e4+b5*e5+b6*e6',
  '[e1,e4]=d1*e1+d2*e2+d3*e3+d4*e4+d5*e5+d6*e6',
  '[e1,e5]= -1*e2',
  '[e1,e6]= 0',

  '[e2,e3]=c1*e1+c2*e2+c3*e3+c4*e4+c5*e5+c6*e6',
  '[e2,e4]=g1*e1+g2*e2+g3*e3+g4*e4+g5*e5+g6*e6',
  '[e2,e5]= 1*e1',
  '[e2,e6]= 0',

  '[e3,e4]=h1*e1+h2*e2+h3*e3+h4*e4+h5*e5+h6*e6',

  '[e3,e5]= 0',
  '[e3,e6]= 1*e4',

  '[e4,e5]= 0',
```

```
  '[e4,e6]= 1*e3',
  '[e5,e6]=0'],
  ['e1','e2','e3','e4','e5','e6'],algx);
```

$LDx := [e1, e2] = a1\ e1 + a2\ e2 + a3\ e3 + a4\ e4 + a5\ e5 + a6\ e6,\ [e1, e3] = b1\ e1 + b2\ e2 + b3\ e3 + b4\ e4 + b5\ e5 + b6\ e6,\ [e1, e4] = d1\ e1 + d2\ e2 + d3\ e3 + d4\ e4 + d5\ e5 + d6\ e6,\ [e1, e5] = -e2,\ [e1, e6] = 0,\ [e2, e3] = c1\ e1 + c2\ e2 + c3\ e3 + c4\ e4 + c5\ e5 + c6\ e6,\ [e2, e4] = g1\ e1 + g2\ e2 + g3\ e3 + g4\ e4 + g5\ e5 + g6\ e6,\ [e2, e5] = e1,\ [e2, e6] = 0,\ [e3, e4] = h1\ e1 + h2\ e2 + h3\ e3 + h4\ e4 + h5\ e5 + h6\ e6,\ [e3, e5] = 0,\ [e3, e6] = e4,\ [e4, e5] = 0,\ [e4, e6] = e3,\ [e5, e6] = 0$ **(1.3)**

```
> DGsetup(LDx, [e], [theta]);
```

Lie algebra: algx **(1.4)**

Extract the linear equations needing solving upon imposing
the Jacobi identities:

```
> ChangeFrame(algx):
  lineqs := []:
  for i from 1 to 6 do
  lineqs||i := []:
   cfs := convert(DGinformation(ExteriorDerivative
  (ExteriorDerivative(theta||i)), "CoefficientSet"), list):
   tf := map(type, cfs, linear):
   cnsts := map(type, cfs, constant):
   if nops(cnsts)>= 1 then
    for l from 1 to nops(cnsts) do
     if cnsts[l] then
      error cnsts[l], "Contradiction, there is a constant
  coefficient.";
     fi;
    od;
   fi;
   for k from 1 to nops(cfs) do
    if tf[k] then
     lineqs||i := [op(lineqs||i), cfs[k]]:
    fi:
   od:
  od:
  for j from 1 to 6 do
   lineqs := [op(lineqs), op(lineqs||j)]:
  od:
  lineqs := convert(lineqs, set):
> lineqs;
```

$\{a2, a3, a4, c3, c4, c5, c6, g3, g4, g5, g6, h2, h3, h4, -a1, -b1, -b2, -b3, -b4, -b5, -b6, -c1, -c2, -c5, -c6, -d1, -d2, -d3, -d4, -d5, -d6, -g1, -g2, -g5, -g6, -h1, -b1 + c2, b2 + c1, -b3 + d4, b3 - d4, b4 - d3, -c1 - b2, -c3 + g4, c3 - g4, c4 - g3, -d1 + g2, d2 + g1, d3 - b4, -g1 - d2, g3 - c4\}$ **(1.5)**

Solve the linear equations:

```
> sol := solve(lineqs);
```

$sol := \{a1 = 0, a2 = 0, a3 = 0, a4 = 0, b1 = 0, b2 = 0, b3 = 0, b4 = 0, b5 = 0, b6 = 0, c1 = 0, c2 = 0, c3 = 0, c4 = 0, c5 = 0, c6 = 0, d1 = 0, d2 = 0, d3 = 0, d4 = 0, d5 = 0, d6 = 0, g1$ **(1.6)**

$= 0, g2 = 0, g3 = 0, g4 = 0, g5 = 0, g6 = 0, h1 = 0, h2 = 0, h3 = 0, h4 = 0\}$

Apply the solutions to the structure equations and setup the
Lie algebra:

```
> LDxx := eval(LDx, sol union {algx=algxx});
```

$$LDxx := [e1, e2] = a5\,e5 + a6\,e6,\ [e1, e3] = 0,\ [e1, e4] = 0,\ [e1, e5] = -e2,\ [e1, e6] = 0,\ [e2, e3] = 0,\ [e2, e4] = 0,\ [e2, e5] = e1,\ [e2, e6] = 0,\ [e3, e4] = h5\,e5 + h6\,e6,\ [e3, e5] = 0,\ [e3, e6] = e4,\ [e4, e5] = 0,\ [e4, e6] = e3,\ [e5, e6] = 0 \quad \textbf{(1.7)}$$

```
> DGsetup(LDxx):
```

Here are the remaining unknowns and new structure equations:

```
> par := indets(LDxx) minus {LDxx, algxx};
```

$$par := \{a5, a6, h5, h6\} \quad \textbf{(1.8)}$$

```
> MultiplicationTable(algxx, "LieTable");
```

algxx	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$a5\,e5 + a6\,e6$	0	0	$-e2$	0
$e2$	$-a5\,e5 - a6\,e6$	0	0	0	$e1$	0
$e3$	0	0	0	$h5\,e5 + h6\,e6$	0	$e4$
$e4$	0	0	$-h5\,e5 - h6\,e6$	0	0	$e3$
$e5$	$e2$	$-e1$	0	0	0	0
$e6$	0	0	$-e4$	$-e3$	0	0

(1.9)

We check the Jacobi identities on these remaining unknowns:
The following exterior derivatives on the 1-forms dual to the
vectors in the Lie algebra are equivalent to the Jacobi identities:

```
> ddtheta1:=ExteriorDerivative(ExteriorDerivative(theta1));
  ddtheta2:=ExteriorDerivative(ExteriorDerivative(theta2));
  ddtheta3:=ExteriorDerivative(ExteriorDerivative(theta3));
  ddtheta4:=ExteriorDerivative(ExteriorDerivative(theta4));
  ddtheta5:=ExteriorDerivative(ExteriorDerivative(theta5));
  ddtheta6:=ExteriorDerivative(ExteriorDerivative(theta6));
```

$$ddtheta1 := -h5\,\theta 2 \wedge \theta 3 \wedge \theta 4$$

$$ddtheta2 := h5\,\theta 1 \wedge \theta 3 \wedge \theta 4$$

$$ddtheta3 := -a6\,\theta 1 \wedge \theta 2 \wedge \theta 4$$

$$ddtheta4 := -a6\,\theta 1 \wedge \theta 2 \wedge \theta 3$$

$$ddtheta5 := 0\,\theta 1 \wedge \theta 2 \wedge \theta 3$$

$$ddtheta6 := 0\,\theta 1 \wedge \theta 2 \wedge \theta 3 \quad \textbf{(1.10)}$$

We see that a6 = 0, h5 = 0. We apply this requirement
to the structure equations and relabel the remaining two unknowns:

```
> ChangeFrame(algxx):
> LD1 := eval(LieAlgebraData([e1,e2,e3,e4,e5,e6],alg1), {a5 = a, a6
  = 0, h5 = 0, h6 = b});
```

$$LD1 := [e1, e2] = a\,e5,\ [e1, e3] = 0,\ [e1, e4] = 0,\ [e1, e5] = -e2,\ [e1, e6] = 0,\ [e2, e3] = 0,\ [e2, e4] = 0,\ [e2, e5] = e1,\ [e2, e6] = 0,\ [e3, e4] = b\,e6,\ [e3, e5] = 0,\ [e3, e6 \quad \textbf{(1.11)}$$

$] = e4,\ [e4, e5] = 0,\ [e4, e6] = e3,\ [e5, e6] = 0$

```
> DGsetup(LD1):
> MultiplicationTable(alg1, "LieTable");
```

alg1	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$a\, e5$	0	0	$-e2$	0
$e2$	$-a\, e5$	0	0	0	$e1$	0
$e3$	0	0	0	$b\, e6$	0	$e4$
$e4$	0	0	$-b\, e6$	0	0	$e3$
$e5$	$e2$	$-e1$	0	0	0	0
$e6$	0	0	$-e4$	$-e3$	0	0

(1.12)

We break into cases on a and b to investigate the nature of the possible Lie algebras. To this end, make the following change of basis:

```
> LD2 := LieAlgebraData([u*e1, u*e2, v*e3, v*e4, e5, e6], alg2);
```

$LD2 := [e1, e2] = u^2\, a\, e5,\ [e1, e3] = 0,\ [e1, e4] = 0,\ [e1, e5] = -e2,\ [e1, e6] = 0,\ [e2, e3] = 0,\ [e2, e4] = 0,\ [e2, e5] = e1,\ [e2, e6] = 0,\ [e3, e4] = v^2\, b\, e6,\ [e3, e5] = 0,\ [e3, e6] = e4,\ [e4, e5] = 0,\ [e4, e6] = e3,\ [e5, e6] = 0$ **(1.13)**

```
> DGsetup(LD2):
> MultiplicationTable(alg2, "LieTable");
```

alg2	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$u^2\, a\, e5$	0	0	$-e2$	0
$e2$	$-u^2\, a\, e5$	0	0	0	$e1$	0
$e3$	0	0	0	$v^2\, b\, e6$	0	$e4$
$e4$	0	0	$-v^2\, b\, e6$	0	0	$e3$
$e5$	$e2$	$-e1$	0	0	0	0
$e6$	0	0	$-e4$	$-e3$	0	0

(1.14)

Pre-Case : a=0, b=0,

If a=0 and b=0, then alg1 is the direct sum of the three dimensional algebra $s_{3,\,3}$, alpha = 0, (in Snobl) with $s_{3,\,1}$, a = -1 (in Snobl). However, this case is flat. Meaning, we compute the adh-invariant inner product on g/h and determine its isometry dimension, which prompts the computation of the curvature tensor which vanishes:

```
> LD4 := eval(LD1, {a=0, b=0, alg1 = alg4});
```

$LD4 := [e1, e2] = 0,\ [e1, e3] = 0,\ [e1, e4] = 0,\ [e1, e5] = -e2,\ [e1, e6] = 0,\ [e2, e3$ **(1.1.1)**

$] = 0,\ [e2, e4] = 0,\ [e2, e5] = e1,\ [e2, e6] = 0,\ [e3, e4] = 0,\ [e3, e5] = 0,\ [e3, e6] = e4,\ [e4, e5] = 0,\ [e4, e6] = e3,\ [e5, e6] = 0$

```
> DGsetup(LD4):
> DGEnvironment[GSpace]([e1,e2,e3,e4], [e5,e6], G, vectorlabels = [X], formlabels = [sigma]);
```

$$G\ Space\colon\ G \tag{1.1.2}$$

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4], 2);
```

$$S := \left[\sigma 1 \otimes \sigma 1,\ \frac{1}{2}\,\sigma 1 \otimes \sigma 2 + \frac{1}{2}\,\sigma 2 \otimes \sigma 1,\ \frac{1}{2}\,\sigma 1 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 1,\ \frac{1}{2}\,\sigma 1 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 1,\ \sigma 2 \otimes \sigma 2,\ \frac{1}{2}\,\sigma 2 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 2,\ \frac{1}{2}\,\sigma 2 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 2,\ \sigma 3 \otimes \sigma 3,\ \frac{1}{2}\,\sigma 3 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 3,\ \sigma 4 \otimes \sigma 4\right] \tag{1.1.3}$$

```
> g := InvariantGeometricObjectFields([X5, X6], S);
```

$$g := \_C1\,\sigma 1 \otimes \sigma 1 + \_C1\,\sigma 2 \otimes \sigma 2 + \_C2\,\sigma 3 \otimes \sigma 3 - \_C2\,\sigma 4 \otimes \sigma 4 \tag{1.1.4}$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$10 \tag{1.1.5}$$

```
> CurvatureTensor(g);
```

$$0\,X1 \otimes \sigma 1 \otimes \sigma 1 \otimes \sigma 1 \tag{1.1.6}$$

Therefore this case is not relevant to out work.

Case 1: a>0, b>0

Let u = 1/sqrt(a) and v = 1/sqrt(b)

```
> ChangeFrame(alg1):
> LD3 := LieAlgebraData([1/sqrt(a)*e1,1/sqrt(a)*e2,1/sqrt(b)*e3,1/sqrt(b)*e4,e5,e6], alg3);
```

$$LD3 := [e1, e2] = e5,\ [e1, e3] = 0,\ [e1, e4] = 0,\ [e1, e5] = -e2,\ [e1, e6] = 0,\ [e2, e3] = 0,\ [e2, e4] = 0,\ [e2, e5] = e1,\ [e2, e6] = 0,\ [e3, e4] = e6,\ [e3, e5] = 0,\ [e3, e6] = e4,\ [e4, e5] = 0,\ [e4, e6] = e3,\ [e5, e6] = 0 \tag{1.15}$$

```
> DGsetup(LD3):
> MultiplicationTable(alg3, "LieTable");
```

(1.16)

alg3	e1	e2	e3	e4	e5	e6
e1	0	e5	0	0	− e2	0
e2	− e5	0	0	0	e1	0
e3	0	0	0	e6	0	e4
e4	0	0	− e6	0	0	e3
e5	e2	− e1	0	0	0	0
e6	0	0	− e4	− e3	0	0

We show this Lie algebra is the direct sum so(3)⊕so(2,1):

```
> LDs3 := SimpleLieAlgebraData("so(3)", so3, version = 1);
```

$$LDs3 := [e1, e2] = e3,\ [e1, e3] = -e2,\ [e2, e3] = e1 \tag{1.17}$$

```
> DGsetup(LDs3):
> LDs21 := SimpleLieAlgebraData("so(2,1)", so21, version = 1);
```

$$LDs21 := [e1, e2] = e2,\ [e1, e3] = -e3,\ [e2, e3] = -e1 \tag{1.18}$$

```
> DGsetup(LDs21):
```

We form the direct sum:

```
> LDs := DirectSum([so3, so21], algs);
```

$$LDs := [e1, e2] = e3,\ [e1, e3] = -e2,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e1, e6] = 0,\ [e2, e3] = e1,\ [e2, e4] = 0,\ [e2, e5] = 0,\ [e2, e6] = 0,\ [e3, e4] = 0,\ [e3, e5] = 0,\ [e3, e6] = 0,\ [e4, e5] = e5,\ [e4, e6] = -e6,\ [e5, e6] = -e4 \tag{1.19}$$

```
> DGsetup(LDs):
```

Here are the structure equations of so(3)⊕so(2,1):

```
> MultiplicationTable(algs, "LieTable");
```

algs	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$e3$	$-e2$	0	0	0
$e2$	$-e3$	0	$e1$	0	0	0
$e3$	$e2$	$-e1$	0	0	0	0
$e4$	0	0	0	0	$e5$	$-e6$
$e5$	0	0	0	$-e5$	0	$-e4$
$e6$	0	0	0	$e6$	$e4$	0

(1.20)

If we make the following change of basis we see it is indeed the direct sum of so(3) and so(2,1):

```
> ChangeFrame(alg3):
> LD3n := LieAlgebraData( [ -e5, e2, e1, e3+e4-e6, -e3-e4, -e4+e6], alg3n);
```

$$LD3n := [e1, e2] = e3,\ [e1, e3] = -e2,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e1, e6] = 0,\ [e2, e3] = e1,\ [e2, e4] = 0,\ [e2, e5] = 0,\ [e2, e6] = 0,\ [e3, e4] = 0,\ [e3, e5] = 0,\ [e3, e6] = 0,\ [e4, e5] = e5,\ [e4, e6] = -e6,\ [e5, e6] = -e4 \tag{1.21}$$

```
> DGsetup(LD3n):
```

The Lie table is now the table for the direct sum of so(3) and so(2,1):

```
> MultiplicationTable(alg3n, "LieTable");
```

alg3n	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$e3$	$-e2$	0	0	0
$e2$	$-e3$	0	$e1$	0	0	0
$e3$	$e2$	$-e1$	0	0	0	0
$e4$	0	0	0	0	$e5$	$-e6$
$e5$	0	0	0	$-e5$	0	$-e4$
$e6$	0	0	0	$e6$	$e4$	0

(1.22)

We check the isometry dimension of the adh-invariant metric on the reductive complement m. First, we initialize the homogeneous space:

```
> ChangeFrame(alg3):
> DGEnvironment[GSpace]([e1,e2,e3,e4], [e5,e6], G, vectorlabels = [X], formlabels = [sigma]);
```

$$G\ Space\colon G \tag{1.23}$$

We need the invariant symmetric rank-2 covariant tensors on m:

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4], 2);
```

$$S := \left[\sigma 1 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 2 + \frac{1}{2}\,\sigma 2 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 1, \sigma 2 \otimes \sigma 2, \frac{1}{2}\,\sigma 2 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 2, \frac{1}{2}\,\sigma 2 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 2, \sigma 3 \otimes \sigma 3, \frac{1}{2}\,\sigma 3 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 3, \sigma 4 \otimes \sigma 4\right] \tag{1.24}$$

The general adh-invariant metric formed from S follows:

```
> g := InvariantGeometricObjectFields([X5, X6], S);
```

$$g := \_C1\,\sigma 1 \otimes \sigma 1 + \_C1\,\sigma 2 \otimes \sigma 2 + \_C2\,\sigma 3 \otimes \sigma 3 - \_C2\,\sigma 4 \otimes \sigma 4 \tag{1.25}$$

The following shows there are no additional symmetries admitted by the metric by computing the symmetry algebra of the metric.
Note the symmetry dimension is six:

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$6 \tag{1.26}$$

Case 1a: a>0, b<0
Let u = 1/sqrt(a) and v = 1/sqrt(-b)
We show this is identical to Case 1.

```
> ChangeFrame(alg1):
> LD3a := LieAlgebraData([1/sqrt(a)*e1,1/sqrt(a)*e2,1/sqrt(-b)*e3, 1/sqrt(-b)*e4,e5,e6], alg3a);
```

$$LD3a := [e1, e2] = e5,\ [e1, e3] = 0,\ [e1, e4] = 0,\ [e1, e5] = -e2,\ [e1, e6] = 0,\ [e2, e3] = 0,\ [e2, e4] = 0,\ [e2, e5] = e1,\ [e2, e6] = 0,\ [e3, e4] = -e6,\ [e3, e5] = 0,\ [e3, e6] = e4,\ [e4, e5] = 0,\ [e4, e6] = e3,\ [e5, e6] = 0 \tag{1.27}$$

```
> DGsetup(LD3a):
> MultiplicationTable(alg3a, "LieTable");
```

(1.28)

alg3a	e1	e2	e3	e4	e5	e6
e1	0	e5	0	0	−e2	0
e2	−e5	0	0	0	e1	0
e3	0	0	0	−e6	0	e4
e4	0	0	e6	0	0	e3
e5	e2	−e1	0	0	0	0
e6	0	0	−e4	−e3	0	0

(1.28)

```
> LDs3 := SimpleLieAlgebraData("so(3)", so3, version = 1);
```

$$LDs3 := [e1, e2] = e3, [e1, e3] = -e2, [e2, e3] = e1 \quad (1.29)$$

```
> DGsetup(LDs3):
> LDs21 := SimpleLieAlgebraData("so(2,1)", so21, version = 1);
```

$$LDs21 := [e1, e2] = e2, [e1, e3] = -e3, [e2, e3] = -e1 \quad (1.30)$$

```
> DGsetup(LDs21):
> LDs := DirectSum([so3, so21], algs);
```

$$LDs := [e1, e2] = e3, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = 0, [e2, e6] = 0, [e3, e4] = 0, [e3, e5] = 0, [e3, e6] = 0, [e4, e5] = e5, [e4, e6] = -e6, [e5, e6] = -e4 \quad (1.31)$$

```
> DGsetup(LDs):
> MultiplicationTable(algs, "LieTable");
```

algs	e1	e2	e3	e4	e5	e6
e1	0	e3	−e2	0	0	0
e2	−e3	0	e1	0	0	0
e3	e2	−e1	0	0	0	0
e4	0	0	0	0	e5	−e6
e5	0	0	0	−e5	0	−e4
e6	0	0	0	e6	e4	0

(1.32)

```
> ChangeFrame(alg3a):
```

This change of basis shows the Lie algebra is so(3)⊕so(2,1), and thus this case is not unique from the previous:

```
> LD3an := LieAlgebraData( [-e5, e1, -e2, e3-e4-e6, e3-e6, e3-e4],
  alg3an);
```

$$LD3an := [e1, e2] = e3, [e1, e3] = -e2, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = 0, [e2, e3] = e1, [e2, e4] = 0, [e2, e5] = 0, [e2, e6] = 0, [e3, e4] = 0, [e3, e5] = 0, [e3, e6] = 0, [e4, e5] = e5, [e4, e6] = -e6, [e5, e6] = -e4 \quad (1.33)$$

```
> DGsetup(LD3an):
> MultiplicationTable(alg3an, "LieTable");
```

alg3an	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$e3$	$-e2$	0	0	0
$e2$	$-e3$	0	$e1$	0	0	0
$e3$	$e2$	$-e1$	0	0	0	0
$e4$	0	0	0	0	$e5$	$-e6$
$e5$	0	0	0	$-e5$	0	$-e4$
$e6$	0	0	0	$e6$	$e4$	0

(1.34)

Case 2: a<0, b>0

Let u = 1/sqrt(-a) and v = 1/sqrt(b)

```
> ChangeFrame(alg1):
> LD4 := LieAlgebraData([1/sqrt(-a)*e1,1/sqrt(-a)*e2,1/sqrt(b)*e3,
  1/sqrt(b)*e4,e5,e6], alg4);
```

$LD4 := [e1, e2] = -e5,\ [e1, e3] = 0,\ [e1, e4] = 0,\ [e1, e5] = -e2,\ [e1, e6] = 0,\ [e2, e3] = 0,\ [e2, e4] = 0,\ [e2, e5] = e1,\ [e2, e6] = 0,\ [e3, e4] = e6,\ [e3, e5] = 0,\ [e3, e6] = e4,\ [e4, e5] = 0,\ [e4, e6] = e3,\ [e5, e6] = 0$ **(1.35)**

```
> DGsetup(LD4):
> MultiplicationTable(alg4, "LieTable");
```

alg4	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$-e5$	0	0	$-e2$	0
$e2$	$e5$	0	0	0	$e1$	0
$e3$	0	0	0	$e6$	0	$e4$
$e4$	0	0	$-e6$	0	0	$e3$
$e5$	$e2$	$-e1$	0	0	0	0
$e6$	0	0	$-e4$	$-e3$	0	0

(1.36)

```
> ChangeFrame(alg4):
```

The following change of basis shows the Lie algebra is so(2,1)⊕so(2,1):

```
> LD4n := LieAlgebraData( [e3-e4-e6, -e4-e6, e3-e4, e1-e2-e5, e1-
  e5, -e2-e5], alg4n);
```

$LD4n := [e1, e2] = e2,\ [e1, e3] = -e3,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e1, e6] = 0,\ [e2, e3] = -e1,\ [e2, e4] = 0,\ [e2, e5] = 0,\ [e2, e6] = 0,\ [e3, e4] = 0,\ [e3, e5] = 0,\ [e3, e6] = 0,\ [e4, e5] = e5,\ [e4, e6] = -e6,\ [e5, e6] = -e4$ **(1.37)**

```
> DGsetup(LD4n):
> MultiplicationTable(alg4n, "LieTable");
```

alg4n	e1	e2	e3	e4	e5	e6
e1	0	e2	− e3	0	0	0
e2	− e2	0	− e1	0	0	0
e3	e3	e1	0	0	0	0
e4	0	0	0	0	e5	− e6
e5	0	0	0	− e5	0	− e4
e6	0	0	0	e6	e4	0

(1.38)

We verify the isometry dimension of the adh-invariant
inner product on g/h:

```
> ChangeFrame(alg4):
> DGEnvironment[GSpace]([e1,e2,e3, e4], [e5,e6], G, vectorlabels =
  [X], formlabels = [sigma]);
```

G Space: G **(1.39)**

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4],
  2);
```

$$S := \left[\sigma 1 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 2 + \frac{1}{2}\,\sigma 2 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 1, \sigma 2 \otimes \sigma 2, \frac{1}{2}\,\sigma 2 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 2, \frac{1}{2}\,\sigma 2 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 2, \sigma 3 \otimes \sigma 3, \frac{1}{2}\,\sigma 3 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 3, \sigma 4 \otimes \sigma 4\right] \quad \textbf{(1.40)}$$

```
> g := InvariantGeometricObjectFields([X5, X6], S);
```

$$g := \_C1\, \sigma 1 \otimes \sigma 1 + \_C1\, \sigma 2 \otimes \sigma 2 + \_C2\, \sigma 3 \otimes \sigma 3 - \_C2\, \sigma 4 \otimes \sigma 4 \quad \textbf{(1.41)}$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

6 **(1.42)**

Case 2a: a<0, b<0
Let u = 1/sqrt(-a) and v = 1/sqrt(-b)
We show this case is identical to the previous.

```
> ChangeFrame(alg1):
> LD4a := LieAlgebraData([1/sqrt(-a)*e1,1/sqrt(-a)*e2,1/sqrt(-b)*
  e3,1/sqrt(-b)*e4,e5,e6], alg4a);
```

$$LD4a := [e1, e2] = -e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e2, [e1, e6] = 0, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = e1, [e2, e6] = 0, [e3, e4] = -e6, [e3, e5] = 0, [e3, e6] = e4, [e4, e5] = 0, [e4, e6] = e3, [e5, e6] = 0 \quad \textbf{(1.43)}$$

```
> DGsetup(LD4a):
> MultiplicationTable(alg4a, "LieTable");
```

alg4a	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$-e5$	0	0	$-e2$	0
$e2$	$e5$	0	0	0	$e1$	0
$e3$	0	0	0	$-e6$	0	$e4$
$e4$	0	0	$e6$	0	0	$e3$
$e5$	$e2$	$-e1$	0	0	0	0
$e6$	0	0	$-e4$	$-e3$	0	0

(1.44)

```
> ChangeFrame(alg4a):
```

This change of basis shows the Lie algebra to be so(2,1)⊕so(2,1):

```
> LD4an := LieAlgebraData( [e1-e2-e5, -e1+e5, e2+e5, -e3+e4+e6, e3-
  e4, e3-e6], alg4an);
```

$LD4an := [e1, e2] = e2,\ [e1, e3] = -e3,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e1, e6] = 0,\ [e2, e3] = -e1,\ [e2, e4] = 0,\ [e2, e5] = 0,\ [e2, e6] = 0,\ [e3, e4] = 0,\ [e3, e5] = 0,\ [e3, e6] = 0,\ [e4, e5] = e5,\ [e4, e6] = -e6,\ [e5, e6] = -e4$ **(1.45)**

```
> DGsetup(LD4an);
```

Lie algebra: alg4an **(1.46)**

```
> MultiplicationTable(alg4an, "LieTable");
```

alg4an	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$e2$	$-e3$	0	0	0
$e2$	$-e2$	0	$-e1$	0	0	0
$e3$	$e3$	$e1$	0	0	0	0
$e4$	0	0	0	0	$e5$	$-e6$
$e5$	0	0	0	$-e5$	0	$-e4$
$e6$	0	0	0	$e6$	$e4$	0

(1.47)

Therefore this case is that of the previous.

Case 3: a=0, b>0

Let v = 1/sqrt(b)

```
> ChangeFrame(alg1):
> LD5 := eval(LieAlgebraData([e1,e2,1/sqrt(b)*e3,1/sqrt(b)*e4,e5,
  e6], alg5), {a=0});
```

$LD5 := [e1, e2] = 0,\ [e1, e3] = 0,\ [e1, e4] = 0,\ [e1, e5] = -e2,\ [e1, e6] = 0,\ [e2, e3] = 0,\ [e2, e4] = 0,\ [e2, e5] = e1,\ [e2, e6] = 0,\ [e3, e4] = e6,\ [e3, e5] = 0,\ [e3, e6] = e4,\ [e4, e5] = 0,\ [e4, e6] = e3,\ [e5, e6] = 0$ **(1.48)**

```
> DGsetup(LD5):
> MultiplicationTable(alg5, "LieTable");
```

alg5	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	0	0	0	$-e2$	0
$e2$	0	0	0	0	$e1$	0
$e3$	0	0	0	$e6$	0	$e4$
$e4$	0	0	$-e6$	0	0	$e3$
$e5$	$e2$	$-e1$	0	0	0	0
$e6$	0	0	$-e4$	$-e3$	0	0

(1.49)

The Lie algebra decomposes as the direct sum
of two three-dimensional subalgebras:

```
> Decompose(alg5);
```

$$\left[\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, [e1, e2, e5, e3, e4, e6]\right] \qquad \textbf{(1.50)}$$

We will show the direct sum is comprised of
$s_{3,\,3}$, alpha = 0, (in Snobl) and so(2,1).
Here is the three dimensional algebra $s_{3,\,3}$, alpha = 0 :

```
> LDWa := Retrieve("Snobl", 1, ["s", 3, 3], algW);
```

$$LDWa := [e1, e2] = 0,\ [e1, e3] = -\_\alpha\, e1 + e2,\ [e2, e3] = -e1 - \_\alpha\, e2 \qquad \textbf{(1.51)}$$

```
> LDW := eval(LDWa, {_alpha=0});
```

$$LDW := [e1, e2] = 0,\ [e1, e3] = e2,\ [e2, e3] = -e1 \qquad \textbf{(1.52)}$$

```
> DGsetup(LDW);
```

$$\textit{Lie algebra: algW} \qquad \textbf{(1.53)}$$

Here we initialize the so(2,1)

```
> LDs21 := SimpleLieAlgebraData("so(2,1)", so21, version = 1);
```

$$LDs21 := [e1, e2] = e2,\ [e1, e3] = -e3,\ [e2, e3] = -e1 \qquad \textbf{(1.54)}$$

```
> DGsetup(LDs21):
```

Initialize the direct sum:

```
> LDs := DirectSum([algW, so21], algs);
```

$$LDs := [e1, e2] = 0,\ [e1, e3] = e2,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e1, e6] = 0,\ [e2, e3] = -e1,\ [e2, e4] = 0,\ [e2, e5] = 0,\ [e2, e6] = 0,\ [e3, e4] = 0,\ [e3, e5] = 0,\ [e3, e6] = 0,\ [e4, e5] = e5,\ [e4, e6] = -e6,\ [e5, e6] = -e4 \qquad \textbf{(1.55)}$$

```
> DGsetup(LDs):
```

Here are the structure equations for the direct sum:

```
> MultiplicationTable(algs, "LieTable");
```

algs	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	0	$e2$	0	0	0
$e2$	0	0	$-e1$	0	0	0
$e3$	$-e2$	$e1$	0	0	0	0
$e4$	0	0	0	0	$e5$	$-e6$
$e5$	0	0	0	$-e5$	0	$-e4$
$e6$	0	0	0	$e6$	$e4$	0

(1.56)

The following change of basis for our Lie algebra shows the correct classification is indeed the direct sum of $s_{3,3}$, alpha = 0 with so(2,1):

```
> ChangeFrame(alg5):
> LD5n := LieAlgebraData( [-e1, -e2, e1+e2-e5, -e6, -(1/2)*e3-(1/2)
  *e4, e3-e4], alg5n);
```

$$LD5n := [e1, e2] = 0, [e1, e3] = e2, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = 0, [e2, e3] = -e1, [e2, e4] = 0, [e2, e5] = 0, [e2, e6] = 0, [e3, e4] = 0, [e3, e5] = 0, [e3, e6] = 0, [e4, e5] = e5, [e4, e6] = -e6, [e5, e6] = -e4 \quad \textbf{(1.57)}$$

```
> DGsetup(LD5n):
> MultiplicationTable(alg5n, "LieTable");
```

alg5n	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	0	$e2$	0	0	0
$e2$	0	0	$-e1$	0	0	0
$e3$	$-e2$	$e1$	0	0	0	0
$e4$	0	0	0	0	$e5$	$-e6$
$e5$	0	0	0	$-e5$	0	$-e4$
$e6$	0	0	0	$e6$	$e4$	0

(1.58)

We check the isometry dimension of the adh-invariant metric on the reductive complement m. First, we initialize the homogeneous space:

```
> ChangeFrame(alg5):
> DGEnvironment[GSpace]([e1,e2,e3, e4], [e5,e6], G, vectorlabels =
  [X], formlabels = [sigma]);
```

$$G\ Space\colon G \quad \textbf{(1.59)}$$

We need the invariant symmetric rank-2 covariant tensors on m:

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4],
  2);
```

$$S := \left[\sigma 1 \otimes \sigma 1,\ \frac{1}{2}\,\sigma 1 \otimes \sigma 2 + \frac{1}{2}\,\sigma 2 \otimes \sigma 1,\ \frac{1}{2}\,\sigma 1 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 1,\ \frac{1}{2}\,\sigma 1 \otimes \sigma 4\right. \quad \textbf{(1.60)}$$

$$+\frac{1}{2}\,\sigma4\otimes\sigma1,\ \sigma2\otimes\sigma2,\ \frac{1}{2}\,\sigma2\otimes\sigma3+\frac{1}{2}\,\sigma3\otimes\sigma2,\ \frac{1}{2}\,\sigma2\otimes\sigma4+\frac{1}{2}\,\sigma4\otimes\sigma2,\ \sigma3\otimes\sigma3,\ \frac{1}{2}\,\sigma3\otimes\sigma4+\frac{1}{2}\,\sigma4\otimes\sigma3,\ \sigma4\otimes\sigma4\Big]$$

The general adh-invariant metric formed from S follows:

```
> g := InvariantGeometricObjectFields([X5, X6], S);
```

$$g := \_C1\ \sigma1\otimes\sigma1+\_C1\ \sigma2\otimes\sigma2+\_C2\ \sigma3\otimes\sigma3-\_C2\ \sigma4\otimes\sigma4 \quad \textbf{(1.61)}$$

The following shows there are no additional symmetries admitted by the metric
by computing the symmetry algebra of the metric.
Note the symmetry dimension is six:

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$6 \quad \textbf{(1.62)}$$

Case 3a: a=0, b<0
Let v = 1/sqrt(-b)
We show this case is identical to Case 3.

```
> ChangeFrame(alg1):
> LD5a := eval(LieAlgebraData([e1,e2,1/sqrt(-b)*e3,1/sqrt(-(b))*e4,
  e5,e6], alg5a), {a=0});
```

$$LD5a := [e1,e2]=0,\ [e1,e3]=0,\ [e1,e4]=0,\ [e1,e5]=-e2,\ [e1,e6]=0,\ [e2,e3]=0,\ [e2,e4]=0,\ [e2,e5]=e1,\ [e2,e6]=0,\ [e3,e4]=-e6,\ [e3,e5]=0,\ [e3,e6]=e4,\ [e4,e5]=0,\ [e4,e6]=e3,\ [e5,e6]=0 \quad \textbf{(1.63)}$$

```
> DGsetup(LD5a):
> MultiplicationTable(alg5a, "LieTable");
```

alg5a	*e1*	*e2*	*e3*	*e4*	*e5*	*e6*
e1	0	0	0	0	− *e2*	0
e2	0	0	0	0	*e1*	0
e3	0	0	0	− *e6*	0	*e4*
e4	0	0	*e6*	0	0	*e3*
e5	*e2*	− *e1*	0	0	0	0
e6	0	0	− *e4*	− *e3*	0	0

(1.64)

Here is the three dimensional algebra $s_{3,\,3}$, alpha = 0 (Snobl)

```
> LDW := LieAlgebraData(['[e1,e2]=0', '[e3,e1]=-e2', '[e3,e2]=e1'],
  ['e1', 'e2', 'e3'], algW);
```

$$LDW := [e1,e2]=0,\ [e1,e3]=e2,\ [e2,e3]=-e1 \quad \textbf{(1.65)}$$

```
> DGsetup(LDW):
> LDs3 := SimpleLieAlgebraData("so(3)", so3, version = 1);
```

$$LDs3 := [e1,e2]=e3,\ [e1,e3]=-e2,\ [e2,e3]=e1 \quad \textbf{(1.66)}$$

```
> DGsetup(LDs3):
> LDs21 := SimpleLieAlgebraData("so(2,1)", so21, version = 1);
```

$$LDs21 := [e1,e2]=e2,\ [e1,e3]=-e3,\ [e2,e3]=-e1 \quad \textbf{(1.67)}$$

```
> DGsetup(LDs21):
> LDs := DirectSum([algW, so21], algs);
```

$LDs := [e1, e2] = 0, [e1, e3] = e2, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = 0, [e2, e3] = -e1, [e2, e4] = 0, [e2, e5] = 0, [e2, e6] = 0, [e3, e4] = 0, [e3, e5] = 0, [e3, e6] = 0, [e4, e5] = e5, [e4, e6] = -e6, [e5, e6] = -e4$ **(1.68)**

```
> DGsetup(LDs):
> MultiplicationTable(algs, "LieTable");
```

algs	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	0	$e2$	0	0	0
$e2$	0	0	$-e1$	0	0	0
$e3$	$-e2$	$e1$	0	0	0	0
$e4$	0	0	0	0	$e5$	$-e6$
$e5$	0	0	0	$-e5$	0	$-e4$
$e6$	0	0	0	$e6$	$e4$	0

(1.69)

The following change of basis shows this is indeed identical to the previous case:

```
> ChangeFrame(alg5a):
> LD5an := LieAlgebraData( [-e1, -e2, e1+e2-e5, -e6, (1/2)*e3+(1/2)
  *e4, e3-e4], alg5an);
```

$LD5an := [e1, e2] = 0, [e1, e3] = e2, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = 0, [e2, e3] = -e1, [e2, e4] = 0, [e2, e5] = 0, [e2, e6] = 0, [e3, e4] = 0, [e3, e5] = 0, [e3, e6] = 0, [e4, e5] = e5, [e4, e6] = -e6, [e5, e6] = -e4$ **(1.70)**

```
> DGsetup(LD5an):
> MultiplicationTable(alg5an, "LieTable");
```

alg5an	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	0	$e2$	0	0	0
$e2$	0	0	$-e1$	0	0	0
$e3$	$-e2$	$e1$	0	0	0	0
$e4$	0	0	0	0	$e5$	$-e6$
$e5$	0	0	0	$-e5$	0	$-e4$
$e6$	0	0	0	$e6$	$e4$	0

(1.71)

Case 4: a>0, b=0

Let u = 1/sqrt(a)

```
> ChangeFrame(alg1):
> LD6 := eval(LieAlgebraData([1/sqrt(a)*e1,1/sqrt(a)*e2,e3,e4,e5,
  e6], alg6), {b=0});
```

$LD6 := [e1, e2] = e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e2, [e1, e6] = 0, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = e1, [e2, e6] = 0, [e3, e4] = 0, [e3, e5] = 0, [e3, e6] = e4, [e4, e5$ **(1.72)**

$] = 0,\ [e4, e6] = e3,\ [e5, e6] = 0$

```
> DGsetup(LD6):
> MultiplicationTable(alg6, "LieTable");
```

$$\begin{array}{c|cccccc} \text{alg6} & e1 & e2 & e3 & e4 & e5 & e6 \\ \hline e1 & 0 & e5 & 0 & 0 & -e2 & 0 \\ e2 & -e5 & 0 & 0 & 0 & e1 & 0 \\ e3 & 0 & 0 & 0 & 0 & 0 & e4 \\ e4 & 0 & 0 & 0 & 0 & 0 & e3 \\ e5 & e2 & -e1 & 0 & 0 & 0 & 0 \\ e6 & 0 & 0 & -e4 & -e3 & 0 & 0 \end{array} \tag{1.73}$$

The Lie algebra decomposes:

```
> Decompose(alg6);
```

$$\left[\left[\begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right], [e1, e2, e5, e3, e4, e6] \right] \tag{1.74}$$

```
> ChangeFrame(alg6):
```

The first three vectors of the decomposition give so(3):

```
> LieAlgebraData( [e1, e2, e5]);
```

$$[e1, e2] = e3,\ [e1, e3] = -e2,\ [e2, e3] = e1 \tag{1.75}$$

The second three vectors give s_3,1, a=-1, from Snobl.

Initialize the second three vectors of the decomposition:

```
> ChangeFrame(alg6):
> DGsetup(LieAlgebraData( [e3, e4, e6], mysno)):
> ChangeFrame(mysno):
```

This change of basis shows it is indeed s_3,1, a=-1, from Snobl:

```
> LieAlgebraData([-e1-e2, -e1+e2, -e2-e3]);
```

$$[e1, e2] = 0,\ [e1, e3] = -e1,\ [e2, e3] = e2 \tag{1.76}$$

Here is s_3,1, a=-1, from Snobl.

```
> LDW := LieAlgebraData(['[e1,e2]=0', '[e3,e1]=e1', '[e3,e2]=-e2'],
  ['e1', 'e2', 'e3'], algW);
```

$$LDW := [e1, e2] = 0,\ [e1, e3] = -e1,\ [e2, e3] = e2 \tag{1.77}$$

```
> DGsetup(LDW):
> LDs3 := SimpleLieAlgebraData("so(3)", so3, version = 1);
```

$$LDs3 := [e1, e2] = e3,\ [e1, e3] = -e2,\ [e2, e3] = e1 \tag{1.78}$$

```
> DGsetup(LDs3):
> LDs := DirectSum([algW,so3], algs);
```

$$LDs := [e1, e2] = 0,\ [e1, e3] = -e1,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e1, e6] = 0,\ [e2, e3] = e2, \tag{1.79}$$

$[e2, e4] = 0,\ [e2, e5] = 0,\ [e2, e6] = 0,\ [e3, e4] = 0,\ [e3, e5] = 0,\ [e3, e6] = 0,\ [e4, e5] = e6,\ [e4, e6] = -e5,\ [e5, e6] = e4$

```
> DGsetup(LDs):
```

Here are the structure equations for the direct sum
s_3,1, a=-1⊕so(3):

```
> MultiplicationTable(algs, "LieTable");
```

algs	e1	e2	e3	e4	e5	e6
e1	0	0	− e1	0	0	0
e2	0	0	e2	0	0	0
e3	e1	− e2	0	0	0	0
e4	0	0	0	0	e6	− e5
e5	0	0	0	− e6	0	e4
e6	0	0	0	e5	− e4	0

(1.80)

```
> ChangeFrame(alg6):
```

This change of basis shows our Lie algebra is
s_3,1, a=-1⊕so(3):

```
> LD6n := LieAlgebraData( [-e3-e4, -e3+e4, -e4-e6, e1,e2,e5],
  alg6n);
```

$LD6n := [e1, e2] = 0,\ [e1, e3] = -e1,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e1, e6] = 0,\ [e2, e3] = e2,\ [e2, e4] = 0,\ [e2, e5] = 0,\ [e2, e6] = 0,\ [e3, e4] = 0,\ [e3, e5] = 0,\ [e3, e6] = 0,\ [e4, e5] = e6,\ [e4, e6] = -e5,\ [e5, e6] = e4$ **(1.81)**

```
> DGsetup(LD6n):
> MultiplicationTable(alg6n, "LieTable");
```

alg6n	e1	e2	e3	e4	e5	e6
e1	0	0	− e1	0	0	0
e2	0	0	e2	0	0	0
e3	e1	− e2	0	0	0	0
e4	0	0	0	0	e6	− e5
e5	0	0	0	− e6	0	e4
e6	0	0	0	e5	− e4	0

(1.82)

We verify the isometry dimension as before:

```
> ChangeFrame(alg6):
> DGEnvironment[GSpace]([e1,e2,e3, e4], [e5,e6], G, vectorlabels =
  [X], formlabels = [sigma]);
```

G Space: G **(1.83)**

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4],
  2);
```

(1.84)

$$S := \left[\sigma 1 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 2 + \frac{1}{2}\,\sigma 2 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 1, \sigma 2 \otimes \sigma 2, \frac{1}{2}\,\sigma 2 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 2, \frac{1}{2}\,\sigma 2 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 2, \sigma 3 \otimes \sigma 3, \frac{1}{2}\,\sigma 3 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 3, \sigma 4 \otimes \sigma 4\right] \quad (1.84)$$

```
> g := InvariantGeometricObjectFields([X5, X6], S);
```

$$g := \_C1\,\sigma 1 \otimes \sigma 1 + \_C1\,\sigma 2 \otimes \sigma 2 + \_C2\,\sigma 3 \otimes \sigma 3 - \_C2\,\sigma 4 \otimes \sigma 4 \quad (1.85)$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$6 \quad (1.86)$$

Case 5: a<0, b=0

Let u = 1/sqrt(-a)

```
> ChangeFrame(alg1):
> LD7 := eval(LieAlgebraData([1/sqrt(-a)*e1,1/sqrt(-a)*e2,e3,e4,e5,
  e6], alg7), {b=0});
```

$$LD7 := [e1, e2] = -e5, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e2, [e1, e6] = 0, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = e1, [e2, e6] = 0, [e3, e4] = 0, [e3, e5] = 0, [e3, e6] = e4, [e4, e5] = 0, [e4, e6] = e3, [e5, e6] = 0 \quad (1.87)$$

```
> DGsetup(LD7):
> MultiplicationTable(alg7, "LieTable");
```

(1.88)

alg7	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	$-e5$	0	0	$-e2$	0
$e2$	$e5$	0	0	0	$e1$	0
$e3$	0	0	0	0	0	$e4$
$e4$	0	0	0	0	0	$e3$
$e5$	$e2$	$-e1$	0	0	0	0
$e6$	0	0	$-e4$	$-e3$	0	0

Note the Lie algebra decomposes:

```
> Decompose(alg7);
```

$$\left[\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, [e3, e4, e6, e1, e2, e5]\right] \quad (1.89)$$

```
> ChangeFrame(alg7):
```

The first three vectors of the decomposition again form This is s_3,1, a=-1, from Snobl.

```
> DGsetup(LieAlgebraData( [e3, e4, e6], mysno)):
> ChangeFrame(mysno):
```

This change of basis on those three vector proves the classification:

```
> LieAlgebraData([-e1-e2, -e1+e2, -e2-e3]);
```

$$[e1, e2] = 0, \ [e1, e3] = -e1, \ [e2, e3] = e2 \quad \textbf{(1.90)}$$

Here is s_3,1 with a=-1 from Snobl:

```
> LDW := LieAlgebraData(['[e1,e2]=0', '[e3,e1]=e1', '[e3,e2]=-e2'],
  ['e1', 'e2', 'e3'], algW);
```

$$LDW := [e1, e2] = 0, \ [e1, e3] = -e1, \ [e2, e3] = e2 \quad \textbf{(1.91)}$$

```
> DGsetup(LDW):
```

Here is so(2,1):

```
> LDs21 := SimpleLieAlgebraData("so(2,1)", so21, version = 1);
```

$$LDs21 := [e1, e2] = e2, \ [e1, e3] = -e3, \ [e2, e3] = -e1 \quad \textbf{(1.92)}$$

```
> DGsetup(LDs21):
```

We form and show the direct sum s_3,1, a=-1⊕so(2,1):

```
> LDs := DirectSum([algW,so21], algs);
```

$$LDs := [e1, e2] = 0, \ [e1, e3] = -e1, \ [e1, e4] = 0, \ [e1, e5] = 0, \ [e1, e6] = 0, \ [e2, e3] = e2, \ [e2, e4] = 0, \ [e2, e5] = 0, \ [e2, e6] = 0, \ [e3, e4] = 0, \ [e3, e5] = 0, \ [e3, e6] = 0, \ [e4, e5] = e5, \ [e4, e6] = -e6, \ [e5, e6] = -e4 \quad \textbf{(1.93)}$$

```
> DGsetup(LDs):
> MultiplicationTable(algs, "LieTable");
```

algs	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	0	$-e1$	0	0	0
$e2$	0	0	$e2$	0	0	0
$e3$	$e1$	$-e2$	0	0	0	0
$e4$	0	0	0	0	$e5$	$-e6$
$e5$	0	0	0	$-e5$	0	$-e4$
$e6$	0	0	0	$e6$	$e4$	0

(1.94)

The following change of basis shows our Lie algebra is s_3,1, a=-1⊕so(2,1):

```
> ChangeFrame(alg7):
> LD7n := LieAlgebraData( [e3+e4, e3-e4, -e3-e6, -e1+e2+e5, e2+e5,
  -e1+e5], alg7n);
```

$$LD7n := [e1, e2] = 0, \ [e1, e3] = -e1, \ [e1, e4] = 0, \ [e1, e5] = 0, \ [e1, e6] = 0, \ [e2, e3] = e2, \ [e2, e4] = 0, \ [e2, e5] = 0, \ [e2, e6] = 0, \ [e3, e4] = 0, \ [e3, e5] = 0, \ [e3, e6] = 0, \ [e4, e5] = e5, \ [e4, e6] = -e6, \ [e5, e6] = -e4 \quad \textbf{(1.95)}$$

```
> DGsetup(LD7n):
> MultiplicationTable(alg7n, "LieTable");
```

alg7n	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	0	$-e1$	0	0	0
$e2$	0	0	$e2$	0	0	0
$e3$	$e1$	$-e2$	0	0	0	0
$e4$	0	0	0	0	$e5$	$-e6$
$e5$	0	0	0	$-e5$	0	$-e4$
$e6$	0	0	0	$e6$	$e4$	0

(1.96)

Check the isometry dimension as before:

```
> ChangeFrame(alg7):
> DGEnvironment[GSpace]([e1,e2,e3, e4], [e5,e6], G, vectorlabels =
  [X], formlabels = [sigma]);
```

$$G\ Space\colon G \tag{1.97}$$

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4],
  2);
```

$$S := \left[\sigma 1 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 2 + \frac{1}{2}\,\sigma 2 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 1, \sigma 2 \otimes \sigma 2, \frac{1}{2}\,\sigma 2 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 2, \frac{1}{2}\,\sigma 2 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 2, \sigma 3 \otimes \sigma 3, \frac{1}{2}\,\sigma 3 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 3, \sigma 4 \otimes \sigma 4\right] \tag{1.98}$$

```
> g := InvariantGeometricObjectFields([X5, X6], S);
```

$$g := \_C1\,\sigma 1 \otimes \sigma 1 + \_C1\,\sigma 2 \otimes \sigma 2 + \_C2\,\sigma 3 \otimes \sigma 3 - \_C2\,\sigma 4 \otimes \sigma 4 \tag{1.99}$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$6 \tag{1.100}$$

This concludes the investigation into the isotropy type F9.

A.4.3 $F10$

Maple Worksheet
Six-dimensional Lie algebra
Two-dimensional Isotropy
Isotropy Type F10

These Maple worksheets aim to validate the claims of chapter 3 regarding the Schmidt method.

F10

A basis for the subalgebra F10 of so(3,1) is the following:

```
> B3,B4;
```

$$\begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad (1.1)$$

The subalgebra is abelian:

```
> B3.B4-B4.B3;
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (1.2)$$

By assuming the above matrices define the adjoint action of e5 and e6 respectively, restricted to a reductive complement m = span({e1, e2, e3, e4}), we obtain the following structure equations:

```
> LDx := LieAlgebraData([
  '[e1,e2]=a1*e1+a2*e2+a3*e3+a4*e4+a5*e5+a6*e6',
  '[e1,e3]=b1*e1+b2*e2+b3*e3+b4*e4+b5*e5+b6*e6',
  '[e1,e4]=d1*e1+d2*e2+d3*e3+d4*e4+d5*e5+d6*e6',
  '[e1,e5]= -e3+e4',
  '[e1,e6]= 0',

  '[e2,e3]=c1*e1+c2*e2+c3*e3+c4*e4+c5*e5+c6*e6',
  '[e2,e4]=g1*e1+g2*e2+g3*e3+g4*e4+g5*e5+g6*e6',
  '[e2,e5]= 0',
  '[e2,e6]= -e3+e4',

  '[e3,e4]=h1*e1+h2*e2+h3*e3+h4*e4+h5*e5+h6*e6',
  '[e3,e5]=e1',
  '[e3,e6]=e2',

  '[e4,e5]=e1',
  '[e4,e6]=e2',
```

```
'[e5,e6]=0'],
['e1','e2','e3','e4','e5','e6'],algx);
```

$LDx := [e1, e2] = a1\ e1 + a2\ e2 + a3\ e3 + a4\ e4 + a5\ e5 + a6\ e6,\ [e1, e3] = b1\ e1 + b2\ e2 + b3\ e3 + b4\ e4 + b5\ e5 + b6\ e6,\ [e1, e4] = d1\ e1 + d2\ e2 + d3\ e3 + d4\ e4 + d5\ e5 + d6\ e6,\ [e1, e5] = -e3 + e4,\ [e1, e6] = 0,\ [e2, e3] = c1\ e1 + c2\ e2 + c3\ e3 + c4\ e4 + c5\ e5 + c6\ e6,\ [e2, e4] = g1\ e1 + g2\ e2 + g3\ e3 + g4\ e4 + g5\ e5 + g6\ e6,\ [e2, e5] = 0,\ [e2, e6] = -e3 + e4,\ [e3, e4] = h1\ e1 + h2\ e2 + h3\ e3 + h4\ e4 + h5\ e5 + h6\ e6,\ [e3, e5] = e1,\ [e3, e6] = e2,\ [e4, e5] = e1,\ [e4, e6] = e2,\ [e5, e6] = 0$ **(1.3)**

Initialize the Lie algebra for use:

```
> DGsetup(LDx, [e], [theta]):
```

Extract the linear equations needing solving upon imposing
the Jacobi identities:

```
> ChangeFrame(algx):
  lineqs := []:
  for i from 1 to 6 do
  lineqs||i := []:
   cfs := convert(DGinformation(ExteriorDerivative
  (ExteriorDerivative(theta||i)), "CoefficientSet"), list):
   tf := map(type, cfs, linear):
   cnsts := map(type, cfs, constant):
   if nops(cnsts)>= 1 then
    for l from 1 to nops(cnsts) do
     if cnsts[l] then
      error cnsts[l], "Contradiction, there is a constant
  coefficient.";
     fi;
    od;
   fi;
   for k from 1 to nops(cfs) do
    if tf[k] then
     lineqs||i := [op(lineqs||i), cfs[k]]:
    fi:
   od:
  od:
  for j from 1 to 6 do
   lineqs := [op(lineqs), op(lineqs||j)]:
  od:
  lineqs := convert(lineqs, set):
```

Here are the linear equations:

```
> lineqs;
```

$\{a2, a5, a6, h1, h2, h5, h6, -a1, -a5, -a6, a3 - c1, a4 + g1, -b1 + h3, b1 + h4, -b2 - a3, b2 - a4, c1 + a4, -c2 + g2, -c2 + h3, c2 + h4, -c5 + g5, -c6 + g6, -d1 + b1, -d1 + h3, d1 + h4, -d2 - a3, -d2 + b2, d2 - a4, -d5 + b5, -d6 + b6, -g1 + a3, -g1 + c1, -g2 + h3, g2 + h4, -g5 + c5, -g6 + c6, -a1 - c3 + g3, a1 + c3 + c4, a1 - c4 + g4, a1 + g3 + g4, -a2 + b3 + b4, -a2 + b3 - d3, -a2 + d3 + d4, a2 + b4 - d4, b3 + b4 + h1, c3 + c4 + h2, d3 + h1 + d4, g3 + h2 + g4, -h1 - d3 + b3, h1 - d4 + b4, -h2 - g3 + c3, h2 - g4 + c4, a3 + b2 + a4 - d2, a3 - c1 + a4 + g1, -d1 + b1 + h3 + h4, -g2 + c2 + h3 + h4\}$ **(1.4)**

We solve for the unknowns:

```
> sol := solve(lineqs);
```

$$sol := \{a1 = 0, a2 = 0, a3 = -a4, a4 = a4, a5 = 0, a6 = 0, b1 = -h4, b2 = a4, b3 = -d4, b4 = d4, b5 = d5, b6 = d6, c1 = -a4, c2 = -h4, c3 = -g4, c4 = g4, c5 = g5, c6 = g6, d1 = -h4, d2 = a4, d3 = -d4, d4 = d4, d5 = d5, d6 = d6, g1 = -a4, g2 = -h4, g3 = -g4, g4 = g4, g5 = g5, g6 = g6, h1 = 0, h2 = 0, h3 = -h4, h4 = h4, h5 = 0, h6 = 0\} \quad \mathbf{(1.5)}$$

Substitute the solution into the structure equations
of the Lie algebra and re-initialize:

```
> LDxx := eval(LDx, sol union {algx=algxx});
```

$$LDxx := [e1, e2] = -a4\, e3 + a4\, e4,\ [e1, e3] = -h4\, e1 + a4\, e2 - d4\, e3 + d4\, e4 + d5\, e5 + d6\, e6,\ [e1, e4] = -h4\, e1 + a4\, e2 - d4\, e3 + d4\, e4 + d5\, e5 + d6\, e6,\ [e1, e5] = -e3 + e4,\ [e1, e6] = 0,\ [e2, e3] = -a4\, e1 - h4\, e2 - g4\, e3 + g4\, e4 + g5\, e5 + g6\, e6,\ [e2, e4] = -a4\, e1 - h4\, e2 - g4\, e3 + g4\, e4 + g5\, e5 + g6\, e6,\ [e2, e5] = 0,\ [e2, e6] = -e3 + e4,\ [e3, e4] = -h4\, e3 + h4\, e4,\ [e3, e5] = e1,\ [e3, e6] = e2,\ [e4, e5] = e1,\ [e4, e6] = e2,\ [e5, e6] = 0 \quad \mathbf{(1.6)}$$

```
> DGsetup(LDxx):
```

Here are the remaining unknowns:

```
> par := indets(LDxx) minus {LDxx, algxx};
```

$$par := \{a4, d4, d5, d6, g4, g5, g6, h4\} \quad \mathbf{(1.7)}$$

We impose the Jacobi identities and determine conditions
on the remaining unknowns:

```
> ddtheta1:=ExteriorDerivative(ExteriorDerivative(theta1));
  ddtheta2:=ExteriorDerivative(ExteriorDerivative(theta2));
  ddtheta3:=ExteriorDerivative(ExteriorDerivative(theta3));
  ddtheta4:=ExteriorDerivative(ExteriorDerivative(theta4));
  ddtheta5:=ExteriorDerivative(ExteriorDerivative(theta5));
  ddtheta6:=ExteriorDerivative(ExteriorDerivative(theta6));
```

$$ddtheta1 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta2 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta3 := -(a4\, h4 + d6 - g5)\ \theta1 \wedge \theta2 \wedge \theta3 - (a4\, h4 + d6 - g5)\ \theta1 \wedge \theta2 \wedge \theta4$$

$$ddtheta4 := (a4\, h4 + d6 - g5)\ \theta1 \wedge \theta2 \wedge \theta3 + (a4\, h4 + d6 - g5)\ \theta1 \wedge \theta2 \wedge \theta4$$

$$ddtheta5 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta6 := 0\ \theta1 \wedge \theta2 \wedge \theta3 \quad \mathbf{(1.8)}$$

```
> ChangeFrame(algxx):
```

Therefore d6=g5-a4*h4:

```
> LD1 := eval(LieAlgebraData([e1,e2,e3,e4,e5,e6], alg1), {d6=g5-a4*
  h4});
```

$$LD1 := [e1, e2] = -a4\, e3 + a4\, e4,\ [e1, e3] = -h4\, e1 + a4\, e2 - d4\, e3 + d4\, e4 + d5\, e5 - (a4\, h4 - g5)\, e6,\ [e1, e4] = -h4\, e1 + a4\, e2 - d4\, e3 + d4\, e4 + d5\, e5 - (a4\, h4 - g5)\, e6,\ [e1, e5] = -e3 + e4,\ [e1, e6] = 0,\ [e2, e3] = -a4\, e1 - h4\, e2 - g4\, e3 + g4\, e4 + g5\, e5 + g6\, e6,\ [e2, e4] = -a4\, e1 - h4\, e2 - g4\, e3 + g4\, e4 + g5\, e5 + g6\, e6,\ [e2, e5] = 0,\ [e2, e6] = -e3 + e4,\ [e3, e4] = -h4\, e3 + h4\, e4,\ [e3, e5 \quad \mathbf{(1.9)}$$

$] = e1, \ [e3, e6] = e2, \ [e4, e5] = e1, \ [e4, e6] = e2, \ [e5, e6] = 0$

```
> DGsetup(LD1):
```

At this stage will refrain from showing the multiplication table. Note the Jacobi identities are satisfied:

```
> Query(alg1, "Jacobi");
```

$$true \qquad (1.10)$$

Here are the remaining unknowns in the structure equations:

```
> par := indets(LD1) minus {LD1, alg1};
```

$$par := \{a4, d4, d5, g4, g5, g6, h4\} \qquad (1.11)$$

Note for generic parameter values, the Lie algebra is solvable and indecomposable:

```
> Query(alg1, "Solvable");
```

$$true \qquad (1.12)$$

```
> Query(alg1, "Indecomposable");
```

$$true \qquad (1.13)$$

```
> ChangeFrame(alg1):
```

We will change basis into a form that puts on display the nilradical. This will show that all cases can be found among s_6,158 through s_6,182 of Snobl. We wish to keep track of the isotropy as we do change basis so find the components of the new basis in terms of the original:

```
> GC1 := GetComponents([e3-e4, e5, e6, e1+a4*e6, e2, e3 + (e3-e4) -
  d4*e5 - g4*e6], [e1,e2,e3,e4,e5,e6]);
```

$$GC1 := [[0, 0, 1, -1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1], [1, 0, 0, 0, 0, a4], [0, 1, 0, 0, 0, 0], [0, 0, 2, -1, -d4, -g4]] \qquad (1.14)$$

The list above defines a matrix which defines the linear transformation giving the change of basis.

```
> Aa := convert(GC1, Matrix)^+;
```

$$Aa := \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -d4 \\ 0 & 0 & 1 & a4 & 0 & -g4 \end{bmatrix} \qquad (1.15)$$

Here we make the change of basis and re-initialize:

```
> ChangeFrame(alg1):
> LD2 := LieAlgebraData([e3-e4, e5, e6, e1+a4*e6, e2, e3 + (e3-e4)
  - d4*e5 - g4*e6], alg2);
```

$$LD2 := [e1, e2] = 0, \ [e1, e3] = 0, \ [e1, e4] = 0, \ [e1, e5] = 0, \ [e1, e6] = -h4\, e1, \ [e2, e3] = 0, \ [e2, e4] = e1, \ [e2, e5] = 0, \ [e2, e6] = a4\, e3 - e4, \ [e3, e4] = 0, \ [e3, e5] = e1, \qquad (1.16)$$

$[e3, e6] = -e5,\ [e4, e5] = 0,\ [e4, e6] = d5\ e2 + g5\ e3 - h4\ e4,\ [e5, e6] = g5\ e2 + (a4^2 + g6)\ e3 - a4\ e4 - h4\ e5$

```
> DGsetup(LD2):
```

We create a linear transformation using the matrix Aa above:

```
> psi := LinearTransformation(alg1, alg2, Aa^(-1));
```

$$\psi := e1 \rightarrow -a4\ e3 + e4,\ e2 \rightarrow e5,\ e3 \rightarrow -e1 + d4\ e2 + g4\ e3 + e6,\ e4 \rightarrow -2\ e1 + d4\ e2 + g4\ e3 + e6,\ e5 \rightarrow e2,\ e6 \rightarrow e3 \quad (1.17)$$

```
> indets(Aa);
```

$$\{a4, d4, g4\} \quad (1.18)$$

Note since phi has parameters in its definition, we wish to verify it is a homomorphism:

```
> Query(alg1, alg2, Aa^(-1), {a4, d4, g4}, "Homomorphism");
```

$$true,\ \{0\},\ [\{a4 = a4, d4 = d4, g4 = g4\}],\ \left[\left[\begin{array}{cccccc} 0 & 0 & -1 & -2 & 0 & 0 \\ 0 & 0 & d4 & d4 & 1 & 0 \\ -a4 & 0 & g4 & g4 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]\right] \quad (1.19)$$

```
> ChangeFrame(alg1):
```

Here is the isotropy in the new algebra:

```
> isoa := ApplyLinearTransformation((psi), [e5,e6]);
```

$$isoa := [e2, e3] \quad (1.20)$$

And we find a reductive complement.
Here a general complement:

```
> cba := ComplementaryBasis(isoa, t);
```

$$cba := [e1 + t1\ e2 + t2\ e3,\ t3\ e2 + t4\ e3 + e4,\ t5\ e2 + t6\ e3 + e5,\ t7\ e2 + t8\ e3 + e6],\ \{t1, t2, t3, t4, t5, t6, t7, t8\} \quad (1.21)$$

We wish to find values of the t's which make the complement reductive:

```
> Query(isoa, cba, "ReductivePair");
```

$$true,\ \{0, t1, t2, -t3, -t5, -t6, -a4 - t4\},\ [\{t1 = 0, t2 = 0, t3 = 0, t4 = -a4, t5 = 0, t6 = 0, t7 = t7, t8 = t8\}],\ [[[e2, e3], [e1, -a4\ e3 + e4, e5, t7\ e2 + t8\ e3 + e6]]] \quad (1.22)$$

Therefore the following is a reductive complement to the isotropy:

```
> compa := [e1, -a4*e3+e4, e5, e6];
```

$$compa := [e1, -a4\ e3 + e4, e5, e6] \quad (1.23)$$

We compute the adjoint representation of the isotropy restricted to the reductive complement and give the isotropy type as a sanity check:

```
> rep := map(Adjoint, isoa, compa);
```

$$rep := \left[\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right] \quad \textbf{(1.24)}$$

Indeed it is of type F10 as it should be:

```
> IsotropyType(rep);
```

"F10" **(1.25)**

We make a relabeling for a more readable table:

```
> LDx := eval(LD2, {a4^2+g6=f, h4=a, a4=b, d5=c, g5=d, alg2=algx});
```

$$LDx := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = -a\, e1, [e2, e3] = 0, [e2, e4] = e1, [e2, e5] = 0, [e2, e6] = b\, e3 - e4, [e3, e4] = 0, [e3, e5] = e1, [e3, e6] = -e5, [e4, e5] = 0, [e4, e6] = c\, e2 + d\, e3 - a\, e4, [e5, e6] = d\, e2 + f\, e3 - b\, e4 - a\, e5 \quad \textbf{(1.26)}$$

```
> DGsetup(LDx):
```

Note the nilradical:

```
> Nilradical(algx);
```

$$[e1, e2, e3, e4, e5] \quad \textbf{(1.27)}$$

Therefore the nilradical is $n_{5,\,3}$ in Snobl. Then the cases are found among $s_{6,\,158}$ through $s_{6,\,182}$ in Snobl.
This concludes the investigation into the isotropy type F10.

A.5 Maple worksheet for G_7 on V_4

A.5.1 $F3$

Maple Worksheet
Seven-dimensional Lie algebra
Three-dimensional Isotropy
Isotropy Type F3

These Maple worksheets aim to validate the claims of chapter 3 regarding the Schmidt method.

These Maple worksheets aim to validate the claims of chapter 3 regarding the Schmidt method.

Here are two bases of so(3,1):

```
> Rx, Ry, Rz, Kx, Ky, Kz:=op(IsotropyType(output="SO31I"));
```

$$Rx, Ry, Rz, Kx, Ky, Kz := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (1)$$

```
> B1,B2,B3,B4,B5,B6:=op(IsotropyType(output="SO31II"));
```

$$B1, B2, B3, B4, B5, B6 := \begin{bmatrix} 0 & -2 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (2)$$

The following are the three-dimensional subalgebras of so(3,1):

F3: {Rx, Ry, Rz} (so(3))
F4: {Rz, Kx, Ky} (so(2,1))
F5: {B(theta),B3, B4}
F6: {B1, B3, B4}
F7: {B2, B3, B4}

F3: {Rx, Ry, Rz}

Here is the basis of F3, which defines so(3) under the matrix commutator:

```
> Rx, Ry, Rz;
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.1}$$

Here are the bracket relations:

```
> Rx.Ry - Ry.Rx; #= e7
```

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.2}$$

```
> Rx.Rz - Rz.Rx; #= -e6
```

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.3}$$

```
> Ry.Rz - Rz.Ry; #= e5
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.4}$$

By assuming the above matrices define the adjoint action of e5, e6, and e7 respectively, restricted to a reductive complement $m = \mathrm{span}(\{e1, e2, e3, e4\})$, we obtain the following structure equations:

```
> LDx := LieAlgebraData([
  '[e1,e2]=a1*e1+a2*e2+a3*e3+a4*e4+a5*e5+a6*e6+a7*e7',
  '[e1,e3]=b1*e1+b2*e2+b3*e3+b4*e4+b5*e5+b6*e6+b7*e7',
  '[e1,e4]=c1*e1+c2*e2+c3*e3+c4*e4+c5*e5+c6*e6+c7*e7',
  '[e1,e5]= 0',
  '[e1,e6]= e3',
  '[e1,e7]= -e2',
  '[e2,e3]=e1*f1+e2*f2+e3*f3+e4*f4+e5*f5+e6*f6+e7*f7',
  '[e2,e4]=e1*g1+e2*g2+e3*g3+e4*g4+e5*g5+e6*g6+e7*g7',
  '[e2,e5]= -e3',
  '[e2,e6]= 0',
  '[e2,e7]= e1',

  '[e3,e4]=e1*h1+e2*h2+e3*h3+e4*h4+e5*h5+e6*h6+e7*h7',
```

```
'[e3,e5]= e2',
'[e3,e6]= -e1',
'[e3,e7]= 0',
'[e4,e5]= 0',
'[e4,e6]= 0',
'[e4,e7]= 0',

'[e5,e6]= e7',
'[e5,e7]= -e6',

'[e6,e7]= e5'],

['e1','e2','e3','e4','e5','e6', 'e7'],algx);
```

$$LDx := [e1, e2] = a1\ e1 + a2\ e2 + a3\ e3 + a4\ e4 + a5\ e5 + a6\ e6 + a7\ e7,\ [e1, e3] = b1\ e1 + b2\ e2 + b3\ e3 + b4\ e4 + b5\ e5 + b6\ e6 + b7\ e7,\ [e1, e4] = c1\ e1 + c2\ e2 + c3\ e3 + c4\ e4 + c5\ e5 + c6\ e6 + c7\ e7,\ [e1, e5] = 0,\ [e1, e6] = e3,\ [e1, e7] = -e2,\ [e2, e3] = f1\ e1 + f2\ e2 + f3\ e3 + f4\ e4 + f5\ e5 + f6\ e6 + f7\ e7,\ [e2, e4] = g1\ e1 + g2\ e2 + g3\ e3 + g4\ e4 + g5\ e5 + g6\ e6 + g7\ e7,\ [e2, e5] = -e3,\ [e2, e6] = 0,\ [e2, e7] = e1,\ [e3, e4] = h1\ e1 + h2\ e2 + h3\ e3 + h4\ e4 + h5\ e5 + h6\ e6 + h7\ e7,\ [e3, e5] = e2,\ [e3, e6] = -e1,\ [e3, e7] = 0,\ [e4, e5] = 0,\ [e4, e6] = 0,\ [e4, e7] = 0,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] = e5 \tag{1.5}$$

Initialize the Lie algebra:

```
> DGsetup(LDx, [e], [theta]):
```

Observe the adjoint representation of the isotropy restricted to the reductive complement:

```
> Adjoint(e5, [e1,e2,e3,e4]), Adjoint(e6, [e1,e2,e3,e4]), Adjoint
  (e7, [e1,e2,e3,e4]);
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.6}$$

Extract the linear equations needing solving upon imposing the Jacobi identities:

```
> ChangeFrame(algx):
  lineqs := []:
  for i from 1 to 7 do
  lineqs||i := []:
   cfs := convert(DGinformation(ExteriorDerivative
  (ExteriorDerivative(theta||i)), "CoefficientSet"), list):
   tf := map(type, cfs, linear):
   cnsts := map(type, cfs, constant):
   if nops(cnsts)>= 1 then
    for l from 1 to nops(cnsts) do
     if cnsts[l] then
      error cnsts[l], "Contradiction, there is a
  constant coefficient.";
     fi;
    od;
```

```
    fi;
    for k from 1 to nops(cfs) do
     if tf[k] then
      lineqs||i := [op(lineqs||i), cfs[k]]:
     fi:
    od:
  od:
  for j from 1 to 7 do
    lineqs := [op(lineqs), op(lineqs||j)]:
  od:
  lineqs := convert(lineqs, set):
```

Here are the linear equations:

```
> lineqs;
```

$$\{a2, a6, b1, b4, b5, c2, c3, c4, c6, c7, f2, f3, f4, f6, f7, g1, g3, g4, g5, g7, h1, h2, h4, h5, h6, -a1, -a2, -a4, -a5, -a6, -b3, -b4, -b7, -c2, -c3, -c4, -c6, -c7, -f2, -f6, -g1, -g3, -g4, -g5, -g7, -h1, -h2, -h4, -h5, -h6, -a1 - f3, a1 + f3, -a2 + b3, -a3 + f1, a3 + b2, -a5 - f7, -a6 + b7, -a7 - b6, -b1 + f2, -b2 - a3, b2 + f1, -b5 + f6, -b6 - f5, b6 + a7, -c1 + g2, c1 - h3, c2 + g1, -c3 - h1, -c5 + g6, c5 - h7, -c6 - g5, c7 + h5, -f1 - b2, f5 - a7, f5 + b6, f7 + a5, -g1 - c2, -g2 + h3, g3 + h2, g5 + c6, -g6 + h7, -g7 - h6, h1 + c3, -h2 - g3, -h5 - c7, h6 + g7\} \tag{1.7}$$

Solve for the unknowns:

```
> sol := solve(lineqs);
```

$$sol := \{a1 = 0, a2 = 0, a3 = -b2, a4 = 0, a5 = 0, a6 = 0, a7 = -b6, b1 = 0, b2 = b2, b3 = 0, b4 = 0, b5 = 0, b6 = b6, b7 = 0, c1 = h3, c2 = 0, c3 = 0, c4 = 0, c5 = h7, c6 = 0, c7 = 0, f1 = -b2, f2 = 0, f3 = 0, f4 = 0, f5 = -b6, f6 = 0, f7 = 0, g1 = 0, g2 = h3, g3 = 0, g4 = 0, g5 = 0, g6 = h7, g7 = 0, h1 = 0, h2 = 0, h3 = h3, h4 = 0, h5 = 0, h6 = 0, h7 = h7\} \tag{1.8}$$

Substitute the solution into the Lie algebra
and re-initialize:

```
> LDxx := eval(LDx, sol union {algx=algxx});
```

$$LDxx := [e1, e2] = -b2\,e3 - b6\,e7,\ [e1, e3] = b2\,e2 + b6\,e6,\ [e1, e4] = h3\,e1 + h7\,e5,\ [e1, e5] = 0,\ [e1, e6] = e3,\ [e1, e7] = -e2,\ [e2, e3] = -b2\,e1 - b6\,e5,\ [e2, e4] = h3\,e2 + h7\,e6,\ [e2, e5] = -e3,\ [e2, e6] = 0,\ [e2, e7] = e1,\ [e3, e4] = h3\,e3 + h7\,e7,\ [e3, e5] = e2,\ [e3, e6] = -e1,\ [e3, e7] = 0,\ [e4, e5] = 0,\ [e4, e6] = 0,\ [e4, e7] = 0,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] = e5 \tag{1.9}$$

```
> DGsetup(LDxx):
```

Here are the remaining unknowns:

```
> par := indets(LDxx) minus {LDxx, algxx};
```

$$par := \{b2, b6, h3, h7\} \tag{1.10}$$

Solving the Jacobi identities for the remaining unknowns
requires we case split as there are quadratics to solve.
Here the command Query shows the two possible
solutions:

```
> H := Query(algxx, par, "Jacobi");
```

$$\tag{1.11}$$

$H := true, \{0, -b2\,h3 + 2\,h7, b2\,h3 - 2\,h7, -b2\,h7 + 2\,b6\,h3, b2\,h7 - 2\,b6\,h3\}, \left[\{b2 = b2, b6 = b6, h3 = 0, h7 = 0\}, \left\{b2 = b2, b6 = \frac{1}{4}\,b2^2, h3 = h3, h7 = \frac{1}{2}\,b2\,h3\right\}\right], \left[[e1, e2] = -b2\,e3 - b6\,e7, [e1, e3] = b2\,e2 + b6\,e6, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = e3, [e1, e7] = -e2, [e2, e3] = -b2\,e1 - b6\,e5, [e2, e4] = 0, [e2, e5] = -e3, [e2, e6] = 0, [e2, e7] = e1, [e3, e4] = 0, [e3, e5] = e2, [e3, e6] = -e1, [e3, e7] = 0, [e4, e5] = 0, [e4, e6] = 0, [e4, e7] = 0, [e5, e6] = e7, [e5, e7] = -e6, [e6, e7] = e5, \left[e1, e2\right] = -b2\,e3 - \frac{b2^2}{4}\,e7, \left[e1, e3\right] = b2\,e2 + \frac{b2^2}{4}\,e6, \left[e1, e4\right] = h3\,e1 + \frac{b2\,h3}{2}\,e5, [e1, e5] = 0, [e1, e6] = e3, [e1, e7] = -e2, \left[e2, e3\right] = -b2\,e1 - \frac{b2^2}{4}\,e5, \left[e2, e4\right] = h3\,e2 + \frac{b2\,h3}{2}\,e6, [e2, e5] = -e3, [e2, e6] = 0, [e2, e7] = e1, \left[e3, e4\right] = h3\,e3 + \frac{b2\,h3}{2}\,e7, [e3, e5] = e2, [e3, e6] = -e1, [e3, e7] = 0, [e4, e5] = 0, [e4, e6] = 0, [e4, e7] = 0, [e5, e6] = e7, [e5, e7] = -e6, [e6, e7] = e5\right]$ **(1.11)**

We now investigate the first solution.

CASE 1:

The command Query named the solutions. Here we rename the Lie algebra of the first solution:

```
> LD1a := eval(H[4][1], "algxx_1"=alg1a);
```

$LD1a := [e1, e2] = -b2\,e3 - b6\,e7, [e1, e3] = b2\,e2 + b6\,e6, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = e3, [e1, e7] = -e2, [e2, e3] = -b2\,e1 - b6\,e5, [e2, e4] = 0, [e2, e5] = -e3, [e2, e6] = 0, [e2, e7] = e1, [e3, e4] = 0, [e3, e5] = e2, [e3, e6] = -e1, [e3, e7] = 0, [e4, e5] = 0, [e4, e6] = 0, [e4, e7] = 0, [e5, e6] = e7, [e5, e7] = -e6, [e6, e7] = e5$ **(1.12)**

```
> DGsetup(LD1a):
> MultiplicationTable(alg1a, "LieTable");
```

(1.13)

alg1a	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$	$e7$
$e1$	0	$-b2\,e3-b6\,e7$	$b2\,e2+b6\,e6$	0	0	$e3$	$-e2$
$e2$	$b2\,e3+b6\,e7$	0	$-b2\,e1-b6\,e5$	0	$-e3$	0	$e1$
$e3$	$-b2\,e2-b6\,e6$	$b2\,e1+b6\,e5$	0	0	$e2$	$-e1$	0
$e4$	0	0	0	0	0	0	0
$e5$	0	$e3$	$-e2$	0	0	$e7$	$-e6$
$e6$	$-e3$	0	$e1$	0	$-e7$	0	$e5$
$e7$	$e2$	$-e1$	0	0	$e6$	$-e5$	0

(1.13)

```
> ChangeFrame(alg1a):
```

Make the following change of basis:

```
> LD1 := LieAlgebraData([e1+(1/2)*b2*e5, e2+(1/2)*b2*e6, e3+(1/2)*
  b2*e7, e4, e5,e6,e7], alg1);
```

$$LD1 := \left[e1, e2\right] = \left(-b6+\frac{b2^2}{4}\right) e7,\ \left[e1, e3\right] = -\left(-b6+\frac{b2^2}{4}\right) e6,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e1, e6] = e3,\ [e1, e7] = -e2,\ \left[e2, e3\right] = \left(-b6+\frac{b2^2}{4}\right) e5,\ [e2, e4] = 0,\ [e2, e5] = -e3,\ [e2, e6] = 0,\ [e2, e7] = e1,\ [e3, e4] = 0,\ [e3, e5] = e2,\ [e3, e6] = -e1,\ [e3, e7] = 0,\ [e4, e5] = 0,\ [e4, e6] = 0,\ [e4, e7] = 0,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] = e5 \quad (1.14)$$

```
> DGsetup(LD1):
```

As a sanity check we compute the Jacobi identities:

```
> ddtheta1:=ExteriorDerivative(ExteriorDerivative(theta1));
  ddtheta2:=ExteriorDerivative(ExteriorDerivative(theta2));
  ddtheta3:=ExteriorDerivative(ExteriorDerivative(theta3));
  ddtheta4:=ExteriorDerivative(ExteriorDerivative(theta4));
  ddtheta5:=ExteriorDerivative(ExteriorDerivative(theta5));
  ddtheta6:=ExteriorDerivative(ExteriorDerivative(theta6));
  ddtheta7:=ExteriorDerivative(ExteriorDerivative(theta7));
```

$$ddtheta1 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta2 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta3 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta4 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta5 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta6 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta7 := 0\ \theta1 \wedge \theta2 \wedge \theta3 \quad (1.15)$$

Reparameterize by letting a = $-b6+\frac{b2^2}{4}$

and re-initialize:

```
> LD2 := eval(LD1, {-b6+(1/4)*b2^2 = a, -(-b6+(1/4)*b2^2) = -a,
```

```
alg1 = alg2});
```

$LD2 := [e1, e2] = a\,e7,\ [e1, e3] = -a\,e6,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e1, e6] = e3,\ [e1, e7] = -e2,\ [e2, e3] = a\,e5,\ [e2, e4] = 0,\ [e2, e5] = -e3,\ [e2, e6] = 0,\ [e2, e7] = e1,\ [e3, e4] = 0,\ [e3, e5] = e2,\ [e3, e6] = -e1,\ [e3, e7] = 0,\ [e4, e5] = 0,\ [e4, e6] = 0,\ [e4, e7] = 0,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] = e5$ **(1.16)**

```
> DGsetup(LD2):
> MultiplicationTable(alg2, "LieTable");
```

alg2	e1	e2	e3	e4	e5	e6	e7
e1	0	$a\,e7$	$-a\,e6$	0	0	$e3$	$-e2$
e2	$-a\,e7$	0	$a\,e5$	0	$-e3$	0	$e1$
e3	$a\,e6$	$-a\,e5$	0	0	$e2$	$-e1$	0
e4	0	0	0	0	0	0	0
e5	0	$e3$	$-e2$	0	0	$e7$	$-e6$
e6	$-e3$	0	$e1$	0	$-e7$	0	$e5$
e7	$e2$	$-e1$	0	0	$e6$	$-e5$	0

(1.17)

Case 1: a>0

```
> ChangeFrame(alg2):
```

Make the following change of basis:

```
> LD3 := LieAlgebraData([e4, 1/sqrt(a)*e1,1/sqrt(a)*e2,1/sqrt(a)*
  e3,e5,e6,e7], alg3);
```

$LD3 := [e1, e2] = 0,\ [e1, e3] = 0,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e1, e6] = 0,\ [e1, e7] = 0,\ [e2, e3] = e7,\ [e2, e4] = -e6,\ [e2, e5] = 0,\ [e2, e6] = e4,\ [e2, e7] = -e3,\ [e3, e4] = e5,\ [e3, e5] = -e4,\ [e3, e6] = 0,\ [e3, e7] = e2,\ [e4, e5] = e3,\ [e4, e6] = -e2,\ [e4, e7] = 0,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] = e5$ **(1.18)**

```
> DGsetup(LD3):
> MultiplicationTable(alg3, "LieTable");
```

alg3	e1	e2	e3	e4	e5	e6	e7
e1	0	0	0	0	0	0	0
e2	0	0	$e7$	$-e6$	0	$e4$	$-e3$
e3	0	$-e7$	0	$e5$	$-e4$	0	$e2$
e4	0	$e6$	$-e5$	0	$e3$	$-e2$	0
e5	0	0	$e4$	$-e3$	0	$e7$	$-e6$
e6	0	$-e4$	0	$e2$	$-e7$	0	$e5$
e7	0	$e3$	$-e2$	0	$e6$	$-e5$	0

(1.19)

```
> ChangeFrame(alg3):
```

The following change of basis shows the Lie algebra is the direct sum so(3)⊕so(3)⊕R:

```
> LD3n := LieAlgebraData( [(1/2)*e4+(1/2)*e7, (1/2)*e3+(1/2)*e6, -(1/2)*e2-(1/2)*e5, (1/2)*e4-(1/2)*e7, (1/2)*e3-(1/2)*e6, (1/2)*e2-(1/2)*e5, -e1], alg3n);
```

$$LD3n := [e1, e2] = e3,\ [e1, e3] = -e2,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e1, e6] = 0,\ [e1, e7] = 0,\ [e2, e3] = e1,\ [e2, e4] = 0,\ [e2, e5] = 0,\ [e2, e6] = 0,\ [e2, e7] = 0,\ [e3, e4] = 0,\ [e3, e5] = 0,\ [e3, e6] = 0,\ [e3, e7] = 0,\ [e4, e5] = e6,\ [e4, e6] = -e5,\ [e4, e7] = 0,\ [e5, e6] = e4,\ [e5, e7] = 0,\ [e6, e7] = 0 \tag{1.20}$$

```
> DGsetup(LD3n):
> MultiplicationTable(alg3n, "LieTable");
```

alg3n	e1	e2	e3	e4	e5	e6	e7
e1	0	e3	− e2	0	0	0	0
e2	− e3	0	e1	0	0	0	0
e3	e2	− e1	0	0	0	0	0
e4	0	0	0	0	e6	− e5	0
e5	0	0	0	− e6	0	e4	0
e6	0	0	0	e5	− e4	0	0
e7	0	0	0	0	0	0	0

(1.21)

```
> ChangeFrame(alg3):
```

We now verify the isometry dimension of the adh-invariant metric on g/h. First define the homogeneous space:

```
> DGEnvironment[GSpace]([e1,e2,e3,e4], [e5,e6,e7], G, vectorlabels = [X], formlabels = [sigma]);
```

$$G\ Space:\ G \tag{1.22}$$

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4], 2);
```

$$S := \left[\sigma1 \otimes \sigma1,\ \frac{1}{2}\,\sigma1 \otimes \sigma2 + \frac{1}{2}\,\sigma2 \otimes \sigma1,\ \frac{1}{2}\,\sigma1 \otimes \sigma3 + \frac{1}{2}\,\sigma3 \otimes \sigma1,\ \frac{1}{2}\,\sigma1 \otimes \sigma4 + \frac{1}{2}\,\sigma4 \otimes \sigma1,\ \sigma2 \otimes \sigma2,\ \frac{1}{2}\,\sigma2 \otimes \sigma3 + \frac{1}{2}\,\sigma3 \otimes \sigma2,\ \frac{1}{2}\,\sigma2 \otimes \sigma4 + \frac{1}{2}\,\sigma4 \otimes \sigma2,\ \sigma3 \otimes \sigma3,\ \frac{1}{2}\,\sigma3 \otimes \sigma4 + \frac{1}{2}\,\sigma4 \otimes \sigma3,\ \sigma4 \otimes \sigma4\right] \tag{1.23}$$

```
> g := InvariantGeometricObjectFields([X5, X6, X7], S);
```

$$g := \_C1\,\sigma1 \otimes \sigma1 + \_C2\,\sigma2 \otimes \sigma2 + \_C2\,\sigma3 \otimes \sigma3 + \_C2\,\sigma4 \otimes \sigma4 \tag{1.24}$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$7 \tag{1.25}$$

Case 2: a<0

```
> ChangeFrame(alg2):
```

Make the following change of basis:

```
> LD4 := LieAlgebraData([e4, 1/sqrt(-a)*e1,1/sqrt(-a)*e2,1/sqrt(-a)
```

```
*e3,e5,e6,e7], alg4);
```

$LD4 := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = 0, [e1, e7] = 0, [e2, e3] = -e7, [e2, e4] = e6, [e2, e5] = 0, [e2, e6] = e4, [e2, e7] = -e3, [e3, e4] = -e5, [e3, e5] = -e4, [e3, e6] = 0, [e3, e7] = e2, [e4, e5] = e3, [e4, e6] = -e2, [e4, e7] = 0, [e5, e6] = e7, [e5, e7] = -e6, [e6, e7] = e5$ **(1.26)**

```
> DGsetup(LD4):
> MultiplicationTable(alg4, "LieTable");
```

alg4	e1	e2	e3	e4	e5	e6	e7
e1	0	0	0	0	0	0	0
e2	0	0	− e7	e6	0	e4	− e3
e3	0	e7	0	− e5	− e4	0	e2
e4	0	− e6	e5	0	e3	− e2	0
e5	0	0	e4	− e3	0	e7	− e6
e6	0	− e4	0	e2	− e7	0	e5
e7	0	e3	− e2	0	e6	− e5	0

(1.27)

```
> ChangeFrame(alg4):
```

This change of basis shows the Lie algebra
to be so(3,1):

```
> LD4n := LieAlgebraData( [e3+e4+e5, e2-e6, -e3-e5, -e2+e7, -e4-e5,
  e2-e6-e7, -e1], alg4n);
```

$LD4n := [e1, e2] = e2, [e1, e3] = e3, [e1, e4] = -e4, [e1, e5] = -e5, [e1, e6] = 0, [e1, e7] = 0, [e2, e3] = 0, [e2, e4] = -e1, [e2, e5] = e6, [e2, e6] = -e3, [e2, e7] = 0, [e3, e4] = -e6, [e3, e5] = -e1, [e3, e6] = e2, [e3, e7] = 0, [e4, e5] = 0, [e4, e6] = -e5, [e4, e7] = 0, [e5, e6] = e4, [e5, e7] = 0, [e6, e7] = 0$ **(1.28)**

```
> DGsetup(LD4n):
> MultiplicationTable(alg4n, "LieTable");
```

alg4n	e1	e2	e3	e4	e5	e6	e7
e1	0	e2	e3	− e4	− e5	0	0
e2	− e2	0	0	− e1	e6	− e3	0
e3	− e3	0	0	− e6	− e1	e2	0
e4	e4	e1	e6	0	0	− e5	0
e5	e5	− e6	e1	0	0	e4	0
e6	0	e3	− e2	e5	− e4	0	0
e7	0	0	0	0	0	0	0

(1.29)

```
> ChangeFrame(alg4):
```

Verify the isometry dimension:

```
> DGEnvironment[GSpace]([e1,e2,e3,e4], [e5,e6,e7], G, vectorlabels
```

```
= [X], formlabels = [sigma]);
```

$$G\ Space:\ G \tag{1.30}$$

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4],
2);
```

$$S := \left[\sigma 1 \otimes \sigma 1, \frac{1}{2}\sigma 1 \otimes \sigma 2 + \frac{1}{2}\sigma 2 \otimes \sigma 1, \frac{1}{2}\sigma 1 \otimes \sigma 3 + \frac{1}{2}\sigma 3 \otimes \sigma 1, \frac{1}{2}\sigma 1 \otimes \sigma 4 + \frac{1}{2}\sigma 4 \otimes \sigma 1, \sigma 2 \otimes \sigma 2, \frac{1}{2}\sigma 2 \otimes \sigma 3 + \frac{1}{2}\sigma 3 \otimes \sigma 2, \frac{1}{2}\sigma 2 \otimes \sigma 4 + \frac{1}{2}\sigma 4 \otimes \sigma 2, \sigma 3 \otimes \sigma 3, \frac{1}{2}\sigma 3 \otimes \sigma 4 + \frac{1}{2}\sigma 4 \otimes \sigma 3, \sigma 4 \otimes \sigma 4\right] \tag{1.31}$$

```
> g := InvariantGeometricObjectFields([X5, X6, X7], S);
```

$$g := \_C1\ \sigma 1 \otimes \sigma 1 + \_C2\ \sigma 2 \otimes \sigma 2 + \_C2\ \sigma 3 \otimes \sigma 3 + \_C2\ \sigma 4 \otimes \sigma 4 \tag{1.32}$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$7 \tag{1.33}$$

Case 3: a=0. Flat metric.

```
> LD5 := eval(LD2, {a=0, alg2=alg5});
```

$$LD5 := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = e3, [e1, e7] = -e2, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = -e3, [e2, e6] = 0, [e2, e7] = e1, [e3, e4] = 0, [e3, e5] = e2, [e3, e6] = -e1, [e3, e7] = 0, [e4, e5] = 0, [e4, e6] = 0, [e4, e7] = 0, [e5, e6] = e7, [e5, e7] = -e6, [e6, e7] = e5 \tag{1.34}$$

```
> DGsetup(LD5):
```

Here we see the Lie algebra decomposes into an
indecomposable six dimensional factor and a
one dimensional factor. We investigate the
six dimensional factor.

```
> Decompose(alg5);
```

$$\left[\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, [e1, e2, e3, e5, e6, e7, e4]\right] \tag{1.35}$$

```
> ChangeFrame(alg5):
```

We initialize the six dimensional

```
> LDsix := LieAlgebraData([ e1, e2, e3, e5, e6, e7], six);
```

$$LDsix := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = e3, [e1, e6] = -e2, [e2, e3] = 0, [e2, e4] = -e3, [e2, e5] = 0, [e2, e6] = e1, [e3, e4] = e2, [e3, e5] = -e1, [e3, e6] = 0, [e4, e5] = e6, [e4, e6] = -e5, [e5, e6] = e4 \tag{1.36}$$

```
> DGsetup(LDsix):
```

```
> MultiplicationTable(six, "LieTable");
```

six	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$
$e1$	0	0	0	0	$e3$	$-e2$
$e2$	0	0	0	$-e3$	0	$e1$
$e3$	0	0	0	$e2$	$-e1$	0
$e4$	0	$e3$	$-e2$	0	$e6$	$-e5$
$e5$	$-e3$	0	$e1$	$-e6$	0	$e4$
$e6$	$e2$	$-e1$	0	$e5$	$-e4$	0

(1.37)

The adh-invariant metric is flat:

```
> ChangeFrame(alg5):
> DGEnvironment[GSpace]([e1,e2,e3,e4], [e5,e6,e7], G, vectorlabels
  = [X], formlabels = [sigma]);
```

$$G\ Space\colon G \tag{1.38}$$

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4],
  2);
```

$$S := \left[\sigma 1 \otimes \sigma 1, \frac{1}{2}\ \sigma 1 \otimes \sigma 2 + \frac{1}{2}\ \sigma 2 \otimes \sigma 1, \frac{1}{2}\ \sigma 1 \otimes \sigma 3 + \frac{1}{2}\ \sigma 3 \otimes \sigma 1, \frac{1}{2}\ \sigma 1 \otimes \sigma 4 + \frac{1}{2}\ \sigma 4 \otimes \sigma 1, \sigma 2 \otimes \sigma 2, \frac{1}{2}\ \sigma 2 \otimes \sigma 3 + \frac{1}{2}\ \sigma 3 \otimes \sigma 2, \frac{1}{2}\ \sigma 2 \otimes \sigma 4 + \frac{1}{2}\ \sigma 4 \otimes \sigma 2, \sigma 3 \otimes \sigma 3, \frac{1}{2}\ \sigma 3 \otimes \sigma 4 + \frac{1}{2}\ \sigma 4 \otimes \sigma 3, \sigma 4 \otimes \sigma 4\right] \tag{1.39}$$

```
> g := InvariantGeometricObjectFields([X5, X6, X7], S);
```

$$g := \_C1\ \sigma 1 \otimes \sigma 1 + \_C1\ \sigma 2 \otimes \sigma 2 + \_C1\ \sigma 3 \otimes \sigma 3 + \_C2\ \sigma 4 \otimes \sigma 4 \tag{1.40}$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$10 \tag{1.41}$$

```
> CurvatureTensor(g);
```

$$0\ X1 \otimes \sigma 1 \otimes \sigma 1 \otimes \sigma 1 \tag{1.42}$$

Therefore this case is excluded from our work.

CASE 2:

These cases will be shown to be excluded.
Again, since command Query named the solutions we
rename the Lie algebra of the second solution:

```
> LD1 := eval(H[4][2], "algxx_2"=alg1);
```

$$LD1 := \left[e1, e2\right] = -b2\ e3 - \frac{b2^2}{4}\ e7, \left[e1, e3\right] = b2\ e2 + \frac{b2^2}{4}\ e6, \left[e1, e4\right] = h3\ e1 + \frac{b2\ h3}{2}\ e5, [e1, e5] = 0, [e1, e6] = e3, [e1, e7] = -e2, \left[e2, e3\right] = -b2\ e1 - \frac{b2^2}{4}\ e5, \left[e2, e4\right] = h3\ e2 + \frac{b2\ h3}{2}\ e6, [e2, e5] = -e3, [e2, e6] = 0, [e2, e7] = e1, \tag{1.43}$$

$[e3, e4] = h3\, e3 + \frac{b2\, h3}{2}\, e7,\ [e3, e5] = e2,\ [e3, e6] = -e1,\ [e3, e7] = 0,\ [e4, e5] = 0,\ [e4, e6] = 0,\ [e4, e7] = 0,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] = e5$

```
> DGsetup(LD1):
> ChangeFrame(alg1):
```

Make the following change of basis:

```
> LD1a := LieAlgebraData( [e1+(1/2)*b2*e5, e2+(1/2)*b2*e6, e3+(1/2)*b2*e7, e4, e5, e6, e7], alg1a);
```

$$LD1a := [e1, e2] = 0,\ [e1, e3] = 0,\ [e1, e4] = h3\, e1,\ [e1, e5] = 0,\ [e1, e6] = e3,\ [e1, e7] = -e2,\ [e2, e3] = 0,\ [e2, e4] = h3\, e2,\ [e2, e5] = -e3,\ [e2, e6] = 0,\ [e2, e7] = e1,\ [e3, e4] = h3\, e3,\ [e3, e5] = e2,\ [e3, e6] = -e1,\ [e3, e7] = 0,\ [e4, e5] = 0,\ [e4, e6] = 0,\ [e4, e7] = 0,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] = e5 \tag{1.44}$$

```
> DGsetup(LD1a);
```

Lie algebra: alg1a **(1.45)**

```
> MultiplicationTable(alg1a, "LieTable");
```

(1.46)

alg1a	e1	e2	e3	e4	e5	e6	e7
e1	0	0	0	h3 e1	0	e3	− e2
e2	0	0	0	h3 e2	− e3	0	e1
e3	0	0	0	h3 e3	e2	− e1	0
e4	− h3 e1	− h3 e2	− h3 e3	0	0	0	0
e5	0	e3	− e2	0	0	e7	− e6
e6	− e3	0	e1	0	− e7	0	e5
e7	e2	− e1	0	0	e6	− e5	0

Subcase 1: h3<>0.
We show that this is a case of constant curvature,
meaning the adh-invariant metric on g/h
has covariantly constant curvature tensor,
and a 10-dimensional isometry algebra:

```
> ChangeFrame(alg1):
> LD1b := eval(LD1a, {alg1a=alg1b});
```

$$LD1b := [e1, e2] = 0,\ [e1, e3] = 0,\ [e1, e4] = h3\, e1,\ [e1, e5] = 0,\ [e1, e6] = e3,\ [e1, e7] = -e2,\ [e2, e3] = 0,\ [e2, e4] = h3\, e2,\ [e2, e5] = -e3,\ [e2, e6] = 0,\ [e2, e7] = e1,\ [e3, e4] = h3\, e3,\ [e3, e5] = e2,\ [e3, e6] = -e1,\ [e3, e7] = 0,\ [e4, e5] = 0,\ [e4, e6] = 0,\ [e4, e7] = 0,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] = e5 \tag{1.47}$$

```
> DGsetup(LD1b):
> DGEnvironment[GSpace]([e1,e2,e3, e4], [e5,e6,e7], G, vectorlabels = [X], formlabels = [sigma]):
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4], 2);
```

$$S := \left[\sigma1 \otimes \sigma1,\ \frac{1}{2}\, \sigma1 \otimes \sigma2 + \frac{1}{2}\, \sigma2 \otimes \sigma1,\ \frac{1}{2}\, \sigma1 \otimes \sigma3 + \frac{1}{2}\, \sigma3 \otimes \sigma1,\ \frac{1}{2}\, \sigma1 \otimes \sigma4 \right. \tag{1.48}$$

$$+\frac{1}{2}\sigma4\otimes\sigma1, \sigma2\otimes\sigma2, \frac{1}{2}\sigma2\otimes\sigma3+\frac{1}{2}\sigma3\otimes\sigma2, \frac{1}{2}\sigma2\otimes\sigma4+\frac{1}{2}\sigma4\otimes\sigma2, \sigma3\otimes\sigma3, \frac{1}{2}\sigma3\otimes\sigma4+\frac{1}{2}\sigma4\otimes\sigma3, \sigma4\otimes\sigma4\Big]$$

```
> g := InvariantGeometricObjectFields([X5, X6, X7], S);
```

$$g := \_C1\,\sigma1\otimes\sigma1+\_C1\,\sigma2\otimes\sigma2+\_C1\,\sigma3\otimes\sigma3+\_C2\,\sigma4\otimes\sigma4 \tag{1.49}$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$10 \tag{1.50}$$

```
> C := CurvatureTensor(g);
```

$$C := -\frac{h3^2\,\_C1}{\_C2}X1\otimes\sigma2\otimes\sigma1\otimes\sigma2+\frac{h3^2\,\_C1}{\_C2}X1\otimes\sigma2\otimes\sigma2\otimes\sigma1-\frac{h3^2\,\_C1}{\_C2}X1\otimes\sigma3\otimes\sigma1\otimes\sigma3+\frac{h3^2\,\_C1}{\_C2}X1\otimes\sigma3\otimes\sigma3\otimes\sigma1-h3^2\,X1\otimes\sigma4\otimes\sigma1\otimes\sigma4+h3^2\,X1\otimes\sigma4\otimes\sigma4\otimes\sigma1+\frac{h3^2\,\_C1}{\_C2}X2\otimes\sigma1\otimes\sigma1\otimes\sigma2-\frac{h3^2\,\_C1}{\_C2}X2\otimes\sigma1\otimes\sigma2\otimes\sigma1-\frac{h3^2\,\_C1}{\_C2}X2\otimes\sigma3\otimes\sigma2\otimes\sigma3+\frac{h3^2\,\_C1}{\_C2}X2\otimes\sigma3\otimes\sigma3\otimes\sigma2-h3^2\,X2\otimes\sigma4\otimes\sigma2\otimes\sigma4+h3^2\,X2\otimes\sigma4\otimes\sigma4\otimes\sigma2+\frac{h3^2\,\_C1}{\_C2}X3\otimes\sigma1\otimes\sigma1\otimes\sigma3-\frac{h3^2\,\_C1}{\_C2}X3\otimes\sigma1\otimes\sigma3\otimes\sigma1+\frac{h3^2\,\_C1}{\_C2}X3\otimes\sigma2\otimes\sigma2\otimes\sigma3-\frac{h3^2\,\_C1}{\_C2}X3\otimes\sigma2\otimes\sigma3\otimes\sigma2-h3^2\,X3\otimes\sigma4\otimes\sigma3\otimes\sigma4+h3^2\,X3\otimes\sigma4\otimes\sigma4\otimes\sigma3+\frac{h3^2\,\_C1}{\_C2}X4\otimes\sigma1\otimes\sigma1\otimes\sigma4-\frac{h3^2\,\_C1}{\_C2}X4\otimes\sigma1\otimes\sigma4\otimes\sigma1+\frac{h3^2\,\_C1}{\_C2}X4\otimes\sigma2\otimes\sigma2\otimes\sigma4-\frac{h3^2\,\_C1}{\_C2}X4\otimes\sigma2\otimes\sigma4\otimes\sigma2+\frac{h3^2\,\_C1}{\_C2}X4\otimes\sigma3\otimes\sigma3\otimes\sigma4-\frac{h3^2\,\_C1}{\_C2}X4\otimes\sigma3\otimes\sigma4\otimes\sigma3 \tag{1.51}$$

```
> CovariantDerivative(C, Christoffel(g) );
```

$$0\,X1\otimes\sigma1\otimes\sigma1\otimes\sigma1\otimes\sigma1 \tag{1.52}$$

Subcase 2: h3=0.
We show that this is a flat case.

```
> ChangeFrame(alg1):
> LD1c := eval(LD1a, {alg1a=alg1c, h3=0});
```

$$LD1c := [e1, e2]=0, [e1, e3]=0, [e1, e4]=0, [e1, e5]=0, [e1, e6]=e3, [e1, e7]=-e2, [e2, e3]=0, [e2, e4]=0, [e2, e5]=-e3, [e2, e6]=0, [e2, e7]=e1, [e3, e4]=0, [e3, e5]=e2, [e3, e6]=-e1, [e3, e7]=0, [e4, e5]=0, [e4, e6]=0, [e4, e7 \tag{1.53}$$

$] = 0,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] = e5$

```
> DGsetup(LD1c);
```

$$\text{Lie algebra: alg1c} \tag{1.54}$$

```
> MultiplicationTable(alg1c, "LieTable");
```

alg1c	e1	e2	e3	e4	e5	e6	e7
e1	0	0	0	0	0	e3	− e2
e2	0	0	0	0	− e3	0	e1
e3	0	0	0	0	e2	− e1	0
e4	0	0	0	0	0	0	0
e5	0	e3	− e2	0	0	e7	− e6
e6	− e3	0	e1	0	− e7	0	e5
e7	e2	− e1	0	0	e6	− e5	0

(1.55)

Setup the homogeneous space:

```
> DGEnvironment[GSpace]([e1,e2,e3, e4], [e5,e6,e7], G, vectorlabels
  = [X], formlabels = [sigma]):
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4],
  2);
```

$$S := \left[\sigma 1 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 2 + \frac{1}{2}\,\sigma 2 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 1, \sigma 2 \otimes \sigma 2, \frac{1}{2}\,\sigma 2 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 2, \frac{1}{2}\,\sigma 2 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 2, \sigma 3 \otimes \sigma 3, \frac{1}{2}\,\sigma 3 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 3, \sigma 4 \otimes \sigma 4\right] \tag{1.56}$$

```
> g := InvariantGeometricObjectFields([X5, X6, X7], S);
```

$$g := \_C1\,\sigma 1 \otimes \sigma 1 + \_C1\,\sigma 2 \otimes \sigma 2 + \_C1\,\sigma 3 \otimes \sigma 3 + \_C2\,\sigma 4 \otimes \sigma 4 \tag{1.57}$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$10 \tag{1.58}$$

```
> CurvatureTensor(g);
```

$$0\,X1 \otimes \sigma 1 \otimes \sigma 1 \otimes \sigma 1 \tag{1.59}$$

A.5.2 $F4$

Maple Worksheet
Seven-dimensional Lie algebra
Three-dimensional Isotropy
Isotropy Type F4

These Maple worksheets aim to validate the claims of chapter 3 regarding the Schmidt method.

F4: {Rz, Kx, Ky}

Here is the basis of F4, which defines so(2,1) under the matrix commutator:

```
> Rz, Kx, Ky;
```

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \mathbf{(1.1)}$$

And the bracket relations, which are those of so(2,1):

```
> Rz.Kx - Kx.Rz; # = e7
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \mathbf{(1.2)}$$

```
> Rz.Ky - Ky.Rz; # = -e6
```

$$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{(1.3)}$$

```
> Kx.Ky - Ky.Kx; # = -e5
```

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{(1.4)}$$

By assuming the above matrices define the adjoint action of e5, e6, and e7 respectively, restricted to a reductive complement m = span({e1, e2, e3, e4}), we obtain the following structure equations:

```
> LDx := LieAlgebraData([
  '[e1,e2]=a1*e1+a2*e2+a3*e3+a4*e4+a5*e5+a6*e6+a7*e7',
  '[e1,e3]=b1*e1+b2*e2+b3*e3+b4*e4+b5*e5+b6*e6+b7*e7',
  '[e1,e4]= c1*e1+c2*e2+c3*e3+c4*e4+c5*e5+c6*e6+c7*e7',
  '[e1,e5]= -e2',
  '[e1,e6]= -e4',
  '[e1,e7]= 0',
  '[e2,e3]=e1*f1+e2*f2+e3*f3+e4*f4+e5*f5+e6*f6+e7*f7',
  '[e2,e4]=e1*g1+e2*g2+e3*g3+e4*g4+e5*g5+e6*g6+e7*g7',
  '[e2,e5]= e1',
  '[e2,e6]= 0',
  '[e2,e7]= -e4',

  '[e3,e4]=e1*h1+e2*h2+e3*h3+e4*h4+e5*h5+e6*h6+e7*h7',
  '[e3,e5]= 0',
  '[e3,e6]= 0',
  '[e3,e7]= 0',
  '[e4,e5]= 0',
  '[e4,e6]= -e1',
  '[e4,e7]= -e2',

  '[e5,e6]= e7',
  '[e5,e7]= -e6',

  '[e6,e7]= -e5'],

  ['e1','e2','e3','e4','e5','e6', 'e7'],algx);
```

$$LDx := [e1, e2] = a1\ e1 + a2\ e2 + a3\ e3 + a4\ e4 + a5\ e5 + a6\ e6 + a7\ e7,\ [e1, e3] = b1\ e1 + b2\ e2 + b3\ e3 + b4\ e4 + b5\ e5 + b6\ e6 + b7\ e7,\ [e1, e4] = c1\ e1 + c2\ e2 + c3\ e3 + c4\ e4 + c5\ e5 + c6\ e6 + c7\ e7,\ [e1, e5] = -e2,\ [e1, e6] = -e4,\ [e1, e7] = 0,\ [e2, e3] = f1\ e1 + f2\ e2 + f3\ e3 + f4\ e4 + f5\ e5 + f6\ e6 + f7\ e7,\ [e2, e4] = g1\ e1 + g2\ e2 + g3\ e3 + g4\ e4 + g5\ e5 + g6\ e6 + g7\ e7,\ [e2, e5] = e1,\ [e2, e6] = 0,\ [e2, e7] = -e4,\ [e3, e4] = h1\ e1 + h2\ e2 + h3\ e3 + h4\ e4 + h5\ e5 + h6\ e6 + h7\ e7,\ [e3, e5] = 0,\ [e3, e6] = 0,\ [e3, e7] = 0,\ [e4, e5] = 0,\ [e4, e6] = -e1,\ [e4, e7] = -e2,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] = -e5 \tag{1.5}$$

Initialize the Lie algebra:

```
> DGsetup(LDx, [e], [theta]):
```

Observe the adjoint representation of the isotropy
restricted to the reductive complement.

```
> Adjoint(e5, [e1,e2,e3,e4]), Adjoint(e6, [e1,e2,e3,e4]), Adjoint
  (e7, [e1,e2,e3,e4]);
```

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{1.6}$$

Extract the linear equations needing solving upon imposing
the Jacobi identities:

```
> ChangeFrame(algx):
```

```
lineqs := []:
for i from 1 to 7 do
lineqs||i := []:
 cfs := convert(DGinformation(ExteriorDerivative
(ExteriorDerivative(theta||i)), "CoefficientSet"), list):
 tf := map(type, cfs, linear):
 cnsts := map(type, cfs, constant):
 if nops(cnsts)>= 1 then
  for l from 1 to nops(cnsts) do
   if cnsts[l] then
    error cnsts[l], "Contradiction, there is a constant
coefficient.";
   fi;
  od;
 fi;
 for k from 1 to nops(cfs) do
  if tf[k] then
   lineqs||i := [op(lineqs||i), cfs[k]]:
  fi:
 od:
od:
for j from 1 to 7 do
 lineqs := [op(lineqs), op(lineqs||j)]:
od:
lineqs := convert(lineqs, set):
```

Here are the linear equations:

```
> lineqs;
```

$$\{a1, a2, a3, a7, c1, c3, c5, c7, f3, f4, f5, f7, g3, g4, g5, h2, h7, -a1, -a2, -a3, -a6, -b2, -b3, -b4, -b5, -b6, -c1, -c3, -c4, -c5, -f1, -f3, -f4, -f7, -g2, -g3, -g4, -g5, -g6, -h1, -h2, -h3, -h6, -h7, -a1 - g4, -a2 + c4, a2 - c4, -a4 + c2, -a4 - g1, -a5 + g7, a5 - c6, a6 - c5, -a7 + g5, -b1 + f2, -b1 - h4, b2 + f1, -b4 - h1, -b5 + h7, -b6 + f7, -b7 - f6, -b7 + h5, -c1 + g2, -c2 + a4, c2 + g1, c5 - a6, -c6 + g7, c6 - a5, -c7 - g6, -f1 - b2, -f2 - h4, -f4 - h2, -f5 - h6, f6 + b7, -g1 - c2, -g5 + a7, g6 + c7, -g7 + a5, -h5 + b7, -h5 - f6, -h7 + b5\} \tag{1.7}$$

Solve for the unknowns:

```
> sol := solve(lineqs);
```

$$sol := \{a1 = 0, a2 = 0, a3 = 0, a4 = c2, a5 = a5, a6 = 0, a7 = 0, b1 = -h4, b2 = 0, b3 = 0, b4 = 0, b5 = 0, b6 = 0, b7 = b7, c1 = 0, c2 = c2, c3 = 0, c4 = 0, c5 = 0, c6 = a5, c7 = 0, f1 = 0, f2 = -h4, f3 = 0, f4 = 0, f5 = 0, f6 = -b7, f7 = 0, g1 = -c2, g2 = 0, g3 = 0, g4 = 0, g5 = 0, g6 = 0, g7 = a5, h1 = 0, h2 = 0, h3 = 0, h4 = h4, h5 = b7, h6 = 0, h7 = 0\} \tag{1.8}$$

Substitute the solutions into the Lie algebra
and initialize:

```
> LDxx := eval(LDx, sol union {algx=algxx});
```

$$LDxx := [e1, e2] = c2\, e4 + a5\, e5,\ [e1, e3] = -h4\, e1 + b7\, e7,\ [e1, e4] = c2\, e2 + a5\, e6,\ [e1, e5] = -e2,\ [e1, e6] = -e4,\ [e1, e7] = 0,\ [e2, e3] = -h4\, e2 - b7\, e6,\ [e2, e4] = -c2\, e1 + a5\, e7,\ [e2, e5] = e1,\ [e2, e6] = 0,\ [e2, e7] = -e4,\ [e3, e4] = h4\, e4 + b7\, e5,\ [e3, e5] = 0,\ [e3, e6] = 0,\ [e3, e7] = 0,\ [e4, e5] = 0,\ [e4, e6] = -e1,\ [e4, e7 \tag{1.9}$$

$] = -e2,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] = -e5$

```
> DGsetup(LDxx):
```

Here are the remaining unknowns:

```
> par := indets(LDxx) minus {LDxx, algxx};
```

$$par := \{a5, b7, c2, h4\} \tag{1.10}$$

We impose the Jacobi identies and note any conditions
on the remaining unknowns:

```
> ddtheta1:=ExteriorDerivative(ExteriorDerivative(theta1));
  ddtheta2:=ExteriorDerivative(ExteriorDerivative(theta2));
  ddtheta3:=ExteriorDerivative(ExteriorDerivative(theta3));
  ddtheta4:=ExteriorDerivative(ExteriorDerivative(theta4));
  ddtheta5:=ExteriorDerivative(ExteriorDerivative(theta5));
  ddtheta6:=ExteriorDerivative(ExteriorDerivative(theta6));
  ddtheta6:=ExteriorDerivative(ExteriorDerivative(theta7));
```

$$ddtheta1 := -\left(-h4\ c2 + 2\ b7\right)\ \theta2 \wedge \theta3 \wedge \theta4$$

$$ddtheta2 := \left(-h4\ c2 + 2\ b7\right)\ \theta1 \wedge \theta3 \wedge \theta4$$

$$ddtheta3 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta4 := -\left(-h4\ c2 + 2\ b7\right)\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta5 := \left(2\ h4\ a5 - c2\ b7\right)\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta6 := -\left(2\ h4\ a5 - c2\ b7\right)\ \theta1 \wedge \theta3 \wedge \theta4$$

$$ddtheta6 := -\left(2\ h4\ a5 - c2\ b7\right)\ \theta2 \wedge \theta3 \wedge \theta4 \tag{1.11}$$

Using the command Query, we see there are
two possible solutions to the above equations
in four unknowns:

```
> H := Query(algxx, par, "Jacobi");
```

$$H := true,\ \{0, -c2\ h4 + 2\ b7, c2\ h4 - 2\ b7, -2\ a5\ h4 + b7\ c2, 2\ a5\ h4 - b7\ c2\},\ \Big[\{a5 = a5, b7 = 0, c2 = c2, h4 = 0\}, \left\{a5 = \frac{1}{4}\ c2^2, b7 = \frac{1}{2}\ h4\ c2, c2 = c2, h4 = h4\right\}\Big],\ \Big[[e1, e2] = c2\ e4 + a5\ e5,\ [e1, e3] = 0,\ [e1, e4] = c2\ e2 + a5\ e6,\ [e1, e5] = -e2,\ [e1, e6] = -e4,\ [e1, e7] = 0,\ [e2, e3] = 0,\ [e2, e4] = -c2\ e1 + a5\ e7,\ [e2, e5] = e1,\ [e2, e6] = 0,\ [e2, e7] = -e4,\ [e3, e4] = 0,\ [e3, e5] = 0,\ [e3, e6] = 0,\ [e3, e7] = 0,\ [e4, e5] = 0,\ [e4, e6] = -e1,\ [e4, e7] = -e2,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] = -e5,\ \Big[e1, e2\Big] = c2\ e4 + \frac{c2^2}{4}\ e5,\ \Big[e1, e3\Big] = -h4\ e1 + \frac{h4\ c2}{2}\ e7,\ \Big[e1, e4\Big] = c2\ e2 + \frac{c2^2}{4}\ e6,\ [e1, e5] = -e2,\ [e1, e6] = -e4,\ [e1, e7] = 0,\ \Big[e2, e3\Big] = -h4\ e2 \tag{1.12}$$

$- \frac{h4\ c2}{2}\ e6,\ \left[e2, e4\right] = -c2\ e1 + \frac{c2^2}{4}\ e7,\ [e2, e5] = e1,\ [e2, e6] = 0,\ [e2, e7] =$
$- e4,\ \left[e3, e4\right] = h4\ e4 + \frac{h4\ c2}{2}\ e5,\ [e3, e5] = 0,\ [e3, e6] = 0,\ [e3, e7] = 0,\ [e4, e5$
$] = 0,\ [e4, e6] = -e1,\ [e4, e7] = -e2,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] =$
$- e5]$

CASE 1:

We rename the Lie algebra of the first solution provided by Query and initialize:

```
> LD1a := eval(H[4][1],"algxx_1"=alg1a );
```

$LD1a := [e1, e2] = c2\ e4 + a5\ e5,\ [e1, e3] = 0,\ [e1, e4] = c2\ e2 + a5\ e6,\ [e1, e5] = -e2,\ [e1, e6] = -e4,\ [e1, e7] = 0,\ [e2, e3] = 0,\ [e2, e4] = -c2\ e1 + a5\ e7,\ [e2, e5] = e1,\ [e2, e6] = 0,\ [e2, e7] = -e4,\ [e3, e4] = 0,\ [e3, e5] = 0,\ [e3, e6] = 0,\ [e3, e7] = 0,\ [e4, e5] = 0,\ [e4, e6] = -e1,\ [e4, e7] = -e2,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] = -e5$ **(1.13)**

```
> DGsetup(LD1a):
> MultiplicationTable(alg1a, "LieTable");
```

alg1a	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$	$e7$
$e1$	0	$c2\ e4 + a5\ e5$	0	$c2\ e2 + a5\ e6$	$-e2$	$-e4$	0
$e2$	$-c2\ e4 - a5\ e5$	0	0	$-c2\ e1 + a5\ e7$	$e1$	0	$-e4$
$e3$	0	0	0	0	0	0	0
$e4$	$-c2\ e2 - a5\ e6$	$c2\ e1 - a5\ e7$	0	0	0	$-e1$	$-e2$
$e5$	$e2$	$-e1$	0	0	0	$e7$	$-e6$
$e6$	$e4$	0	0	$e1$	$-e7$	0	$-e5$
$e7$	0	$e4$	0	$e2$	$e6$	$e5$	0

(1.14)

As a sanity check, we verify the Jacobi identites:

```
> ddtheta1:=ExteriorDerivative(ExteriorDerivative(theta1));
  ddtheta2:=ExteriorDerivative(ExteriorDerivative(theta2));
  ddtheta3:=ExteriorDerivative(ExteriorDerivative(theta3));
  ddtheta4:=ExteriorDerivative(ExteriorDerivative(theta4));
  ddtheta5:=ExteriorDerivative(ExteriorDerivative(theta5));
  ddtheta6:=ExteriorDerivative(ExteriorDerivative(theta6));
  ddtheta7:=ExteriorDerivative(ExteriorDerivative(theta7));
```

$$ddtheta1 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta2 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta3 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta4 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta5 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta6 := 0\ \theta 1 \wedge \theta 2 \wedge \theta 3$$

$$ddtheta7 := 0\ \theta 1 \wedge \theta 2 \wedge \theta 3 \quad \mathbf{(1.15)}$$

```
> ChangeFrame(alg1a):
```

Make the following change of basis:

```
> LD1 := LieAlgebraData([e3, e1-(1/2)*c2*e7, e2+(1/2)*c2*e6, e4+
  (1/2)*c2*e5, e5,e6,e7], alg1);
```

$$LD1 := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = 0, [e1, e7] = 0, \left[e2, e3\right] = \left(-\frac{c2^2}{4} + a5\right) e5, \left[e2, e4\right] = \left(-\frac{c2^2}{4} + a5\right) e6, [e2, e5] = -e3, [e2, e6] = -e4, [e2, e7] = 0, \left[e3, e4\right] = \left(-\frac{c2^2}{4} + a5\right) e7, [e3, e5] = e2, [e3, e6] = 0, [e3, e7] = -e4, [e4, e5] = 0, [e4, e6] = -e2, [e4, e7] = -e3, [e5, e6] = e7, [e5, e7] = -e6, [e6, e7] = -e5 \quad \mathbf{(1.16)}$$

```
> DGsetup(LD1):
```

Reparameterize by letting $a = -\frac{c2^2}{4} + a5$.

Then we case split on a>0, a=0, a<0.

```
> LD2 := eval(LD1, {-(1/4)*c2^2+a5 = a, -(-(1/4)*c2^2+a5) = -a,
  alg1 = alg2});
```

$$LD2 := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = 0, [e1, e7] = 0, [e2, e3] = a\ e5, [e2, e4] = a\ e6, [e2, e5] = -e3, [e2, e6] = -e4, [e2, e7] = 0, [e3, e4] = a\ e7, [e3, e5] = e2, [e3, e6] = 0, [e3, e7] = -e4, [e4, e5] = 0, [e4, e6] = -e2, [e4, e7] = -e3, [e5, e6] = e7, [e5, e7] = -e6, [e6, e7] = -e5 \quad \mathbf{(1.17)}$$

```
> DGsetup(LD2):
> MultiplicationTable(alg2, "LieTable");
```

alg2	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$	$e7$
$e1$	0	0	0	0	0	0	0
$e2$	0	0	$a\ e5$	$a\ e6$	$-e3$	$-e4$	0
$e3$	0	$-a\ e5$	0	$a\ e7$	$e2$	0	$-e4$
$e4$	0	$-a\ e6$	$-a\ e7$	0	0	$-e2$	$-e3$
$e5$	0	$e3$	$-e2$	0	0	$e7$	$-e6$
$e6$	0	$e4$	0	$e2$	$-e7$	0	$-e5$
$e7$	0	0	$e4$	$e3$	$e6$	$e5$	0

(1.18)

Case 1: a>0

```
> ChangeFrame(alg2):
```

Observe the change of basis:

```
> LD3 := LieAlgebraData([e1, 1/sqrt(a)*e2,1/sqrt(a)*e3,1/sqrt(a)*
```

```
e4,e5, e6,e7 ], alg3);
```

$LD3 := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = 0, [e1, e7] = 0, [e2, e3] = e5, [e2, e4] = e6, [e2, e5] = -e3, [e2, e6] = -e4, [e2, e7] = 0, [e3, e4] = e7, [e3, e5] = e2, [e3, e6] = 0, [e3, e7] = -e4, [e4, e5] = 0, [e4, e6] = -e2, [e4, e7] = -e3, [e5, e6] = e7, [e5, e7] = -e6, [e6, e7] = -e5$ **(1.19)**

```
> DGsetup(LD3):
> MultiplicationTable(alg3, "LieTable");
```

alg3	e1	e2	e3	e4	e5	e6	e7
e1	0	0	0	0	0	0	0
e2	0	0	e5	e6	− e3	− e4	0
e3	0	− e5	0	e7	e2	0	− e4
e4	0	− e6	− e7	0	0	− e2	− e3
e5	0	e3	− e2	0	0	e7	− e6
e6	0	e4	0	e2	− e7	0	− e5
e7	0	0	e4	e3	e6	e5	0

(1.20)

```
> ChangeFrame(alg3):
```

This change of basis shows the Lie algebra to be so(3,1)⊕R:

```
> LD3n := LieAlgebraData( [e2, e3, e5, -e4, -e6, -e7, e1], alg3n);
```

$LD3n := [e1, e2] = e3, [e1, e3] = -e2, [e1, e4] = e5, [e1, e5] = -e4, [e1, e6] = 0, [e1, e7] = 0, [e2, e3] = e1, [e2, e4] = e6, [e2, e5] = 0, [e2, e6] = -e4, [e2, e7] = 0, [e3, e4] = 0, [e3, e5] = e6, [e3, e6] = -e5, [e3, e7] = 0, [e4, e5] = -e1, [e4, e6] = -e2, [e4, e7] = 0, [e5, e6] = -e3, [e5, e7] = 0, [e6, e7] = 0$ **(1.21)**

```
> DGsetup(LD3n);
```

Lie algebra: alg3n **(1.22)**

```
> MultiplicationTable(alg3n, "LieTable");
```

alg3n	e1	e2	e3	e4	e5	e6	e7
e1	0	e3	− e2	e5	− e4	0	0
e2	− e3	0	e1	e6	0	− e4	0
e3	e2	− e1	0	0	e6	− e5	0
e4	− e5	− e6	0	0	− e1	− e2	0
e5	e4	0	− e6	e1	0	− e3	0
e6	0	e4	e5	e2	e3	0	0
e7	0	0	0	0	0	0	0

(1.23)

Now we verify the isometry dimension:

```
> ChangeFrame(alg3):
> DGEnvironment[GSpace]([e1,e2,e3,e4], [e5,e6,e7], G, vectorlabels
  = [X], formlabels = [sigma]);
```

$$G\ Space:\ G \tag{1.24}$$

We need the symmetric tensors to construct the general metric:

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4], 2);
```

$$S := \left[\sigma 1 \otimes \sigma 1, \frac{1}{2}\sigma 1 \otimes \sigma 2 + \frac{1}{2}\sigma 2 \otimes \sigma 1, \frac{1}{2}\sigma 1 \otimes \sigma 3 + \frac{1}{2}\sigma 3 \otimes \sigma 1, \frac{1}{2}\sigma 1 \otimes \sigma 4 + \frac{1}{2}\sigma 4 \otimes \sigma 1, \sigma 2 \otimes \sigma 2, \frac{1}{2}\sigma 2 \otimes \sigma 3 + \frac{1}{2}\sigma 3 \otimes \sigma 2, \frac{1}{2}\sigma 2 \otimes \sigma 4 + \frac{1}{2}\sigma 4 \otimes \sigma 2, \sigma 3 \otimes \sigma 3, \frac{1}{2}\sigma 3 \otimes \sigma 4 + \frac{1}{2}\sigma 4 \otimes \sigma 3, \sigma 4 \otimes \sigma 4\right] \tag{1.25}$$

Here is the general metric:

```
> g := InvariantGeometricObjectFields([X5, X6, X7], S);
```

$$g := \_C1\, \sigma 1 \otimes \sigma 1 + \_C2\, \sigma 2 \otimes \sigma 2 + \_C2\, \sigma 3 \otimes \sigma 3 - \_C2\, \sigma 4 \otimes \sigma 4 \tag{1.26}$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$7 \tag{1.27}$$

Therefore, the Lie algebra is the full isometry algebra.

Case 2: a<0

```
> ChangeFrame(alg2):
```

Observe the following change of basis:

```
> LD4 := LieAlgebraData([e1, 1/sqrt(-a)*e2,1/sqrt(-a)*e3,1/sqrt(-a)*e4,e5, e6,e7], alg4);
```

$$LD4 := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = 0, [e1, e7] = 0, [e2, e3] = -e5, [e2, e4] = -e6, [e2, e5] = -e3, [e2, e6] = -e4, [e2, e7] = 0, [e3, e4] = -e7, [e3, e5] = e2, [e3, e6] = 0, [e3, e7] = -e4, [e4, e5] = 0, [e4, e6] = -e2, [e4, e7] = -e3, [e5, e6] = e7, [e5, e7] = -e6, [e6, e7] = -e5 \tag{1.28}$$

```
> DGsetup(LD4):
> MultiplicationTable(alg4, "LieTable");
```

alg4	e1	e2	e3	e4	e5	e6	e7
e1	0	0	0	0	0	0	0
e2	0	0	− e5	− e6	− e3	− e4	0
e3	0	e5	0	− e7	e2	0	− e4
e4	0	e6	e7	0	0	− e2	− e3
e5	0	e3	− e2	0	0	e7	− e6
e6	0	e4	0	e2	− e7	0	− e5
e7	0	0	e4	e3	e6	e5	0

(1.29)

```
> ChangeFrame(alg4):
```

The following change of basis shows the Lie algebra
to be so(2,1)⊕so(2,1)⊕R:

```
> LD4n := LieAlgebraData( [(1/2)*(e4-e5), (1/2)*(e2+e7), -1/2*(e3-
```

```
e6), (1/2)*(e4+e5), (1/2)*(e2-e7), 1/2*(e3+e6), e1], alg4n);
```

$$LD4n := [e1, e2] = e3,\ [e1, e3] = -e2,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e1, e6] = 0,\ [e1, e7] = 0,\ [e2, e3] = -e1,\ [e2, e4] = 0,\ [e2, e5] = 0,\ [e2, e6] = 0,\ [e2, e7] = 0,\ [e3, e4] = 0,\ [e3, e5] = 0,\ [e3, e6] = 0,\ [e3, e7] = 0,\ [e4, e5] = e6,\ [e4, e6] = -e5,\ [e4, e7] = 0,\ [e5, e6] = -e4,\ [e5, e7] = 0,\ [e6, e7] = 0 \tag{1.30}$$

```
> DGsetup(LD4n):
> MultiplicationTable(alg4n, "LieTable");
```

alg4n	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$	$e7$
$e1$	0	$e3$	$-e2$	0	0	0	0
$e2$	$-e3$	0	$-e1$	0	0	0	0
$e3$	$e2$	$e1$	0	0	0	0	0
$e4$	0	0	0	0	$e6$	$-e5$	0
$e5$	0	0	0	$-e6$	0	$-e4$	0
$e6$	0	0	0	$e5$	$e4$	0	0
$e7$	0	0	0	0	0	0	0

(1.31)

```
> ChangeFrame(alg4):
```

Now we verify the isometry dimension:

```
> DGEnvironment[GSpace]([e1,e2,e3,e4], [e5,e6,e7], G, vectorlabels
  = [X], formlabels = [sigma]);
```

$$G\ Space\colon\ G \tag{1.32}$$

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4],
  2);
```

$$S := \left[\sigma 1 \otimes \sigma 1,\ \frac{1}{2}\,\sigma 1 \otimes \sigma 2 + \frac{1}{2}\,\sigma 2 \otimes \sigma 1,\ \frac{1}{2}\,\sigma 1 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 1,\ \frac{1}{2}\,\sigma 1 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 1,\ \sigma 2 \otimes \sigma 2,\ \frac{1}{2}\,\sigma 2 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 2,\ \frac{1}{2}\,\sigma 2 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 2,\ \sigma 3 \otimes \sigma 3,\ \frac{1}{2}\,\sigma 3 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 3,\ \sigma 4 \otimes \sigma 4\right] \tag{1.33}$$

```
> g := InvariantGeometricObjectFields([X5, X6, X7], S);
```

$$g := \_C1\,\sigma 1 \otimes \sigma 1 + \_C2\,\sigma 2 \otimes \sigma 2 + \_C2\,\sigma 3 \otimes \sigma 3 - \_C2\,\sigma 4 \otimes \sigma 4 \tag{1.34}$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$7 \tag{1.35}$$

Case 3: a=0

```
> LD4 := eval(LD2, {a=0, alg2=alg4});
```

$$LD4 := [e1, e2] = 0,\ [e1, e3] = 0,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e1, e6] = 0,\ [e1, e7] = 0,\ [e2, e3] = 0,\ [e2, e4] = 0,\ [e2, e5] = -e3,\ [e2, e6] = -e4,\ [e2, e7] = 0,\ [e3, e4] = 0,\ [e3, e5] = e2,\ [e3, e6] = 0,\ [e3, e7] = -e4,\ [e4, e5] = 0,\ [e4, e6] = -e2,\ [e4, e7] = -e3,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] = -e5 \tag{1.36}$$

```
> DGsetup(LD4):
```

```
> MultiplicationTable(alg4, "LieTable");
```

alg4	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$	$e7$
$e1$	0	0	0	0	0	0	0
$e2$	0	0	0	0	$-e3$	$-e4$	0
$e3$	0	0	0	0	$e2$	0	$-e4$
$e4$	0	0	0	0	0	$-e2$	$-e3$
$e5$	0	$e3$	$-e2$	0	0	$e7$	$-e6$
$e6$	0	$e4$	0	$e2$	$-e7$	0	$-e5$
$e7$	0	0	$e4$	$e3$	$e6$	$e5$	0

(1.37)

The adh-invariant inner product on g/h is flat:

```
> ChangeFrame(alg4):
> DGEnvironment[GSpace]([e1,e2,e3,e4], [e5,e6,e7], G, vectorlabels
  = [X], formlabels = [sigma]);
```

$$G\ Space\!: G \tag{1.38}$$

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4],
  2);
```

$$S := \left[\sigma 1 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 2 + \frac{1}{2}\,\sigma 2 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 1, \sigma 2 \otimes \sigma 2, \frac{1}{2}\,\sigma 2 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 2, \frac{1}{2}\,\sigma 2 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 2, \sigma 3 \otimes \sigma 3, \frac{1}{2}\,\sigma 3 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 3, \sigma 4 \otimes \sigma 4\right] \tag{1.39}$$

```
> g := InvariantGeometricObjectFields([X5, X6, X7], S);
```

$$g := \_C1\,\sigma 1 \otimes \sigma 1 + \_C2\,\sigma 2 \otimes \sigma 2 + \_C2\,\sigma 3 \otimes \sigma 3 - \_C2\,\sigma 4 \otimes \sigma 4 \tag{1.40}$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$10 \tag{1.41}$$

Observe the curvature tensor vanishes:

```
> CurvatureTensor(g);
```

$$0\,X1 \otimes \sigma 1 \otimes \sigma 1 \otimes \sigma 1 \tag{1.42}$$

CASE 2:

We demonstrate that this case is excluded from the classification.

We rename the Lie algebra of the first solution
provided by Query and initialize:

```
> LD1 := eval(H[4][2],"algxx_2"=alg1 );
```

$$LD1 := \left[e1, e2\right] = c2\,e4 + \frac{c2^2}{4}\,e5, \left[e1, e3\right] = -h4\,e1 + \frac{h4\,c2}{2}\,e7, \left[e1, e4\right] = c2\,e2 + \frac{c2^2}{4}\,e6, [e1, e5] = -e2, [e1, e6] = -e4, [e1, e7] = 0, \left[e2, e3\right] = -h4\,e2 \tag{1.43}$$

$-\frac{h4\ c2}{2}\ e6,\ \left[e2, e4\right] = -c2\ e1 + \frac{c2^2}{4}\ e7,\ [e2, e5] = e1,\ [e2, e6] = 0,\ [e2, e7] = -e4,\ \left[e3, e4\right] = h4\ e4 + \frac{h4\ c2}{2}\ e5,\ [e3, e5] = 0,\ [e3, e6] = 0,\ [e3, e7] = 0,\ [e4, e5] = 0,\ [e4, e6] = -e1,\ [e4, e7] = -e2,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] = -e5$

```
> DGsetup(LD1):
```

As a sanity check, we verify the Jacobi identites:

```
> ddtheta1:=ExteriorDerivative(ExteriorDerivative(theta1));
  ddtheta2:=ExteriorDerivative(ExteriorDerivative(theta2));
  ddtheta3:=ExteriorDerivative(ExteriorDerivative(theta3));
  ddtheta4:=ExteriorDerivative(ExteriorDerivative(theta4));
  ddtheta5:=ExteriorDerivative(ExteriorDerivative(theta5));
  ddtheta6:=ExteriorDerivative(ExteriorDerivative(theta6));
  ddtheta7:=ExteriorDerivative(ExteriorDerivative(theta7));
```

$$ddtheta1 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta2 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta3 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta4 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta5 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta6 := 0\ \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta7 := 0\ \theta1 \wedge \theta2 \wedge \theta3 \qquad \textbf{(1.44)}$$

Subcase 1: h4<>0. Constant curvature.

```
> ChangeFrame(alg1):
```

Note the change of basis:

```
> LDs2 := LieAlgebraData([e1-(1/2)*c2*e7, e2+(1/2)*c2*e6, 1/h4*e3,
  e4+(1/2)*c2*e5, e5, e6, e7], salg2);
```

$LDs2 := [e1, e2] = 0,\ [e1, e3] = -e1,\ [e1, e4] = 0,\ [e1, e5] = -e2,\ [e1, e6] = -e4,\ [e1, e7] = 0,\ [e2, e3] = -e2,\ [e2, e4] = 0,\ [e2, e5] = e1,\ [e2, e6] = 0,\ [e2, e7] = -e4,\ [e3, e4] = e4,\ [e3, e5] = 0,\ [e3, e6] = 0,\ [e3, e7] = 0,\ [e4, e5] = 0,\ [e4, e6] = -e1,\ [e4, e7] = -e2,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] = -e5$ **(1.45)**

```
> DGsetup(LDs2):
> MultiplicationTable(salg2, "LieTable");
```

(1.46)

salg2	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$	$e7$
$e1$	0	0	$-e1$	0	$-e2$	$-e4$	0
$e2$	0	0	$-e2$	0	$e1$	0	$-e4$
$e3$	$e1$	$e2$	0	$e4$	0	0	0
$e4$	0	0	$-e4$	0	0	$-e1$	$-e2$
$e5$	$e2$	$-e1$	0	0	0	$e7$	$-e6$
$e6$	$e4$	0	0	$e1$	$-e7$	0	$-e5$
$e7$	0	$e4$	0	$e2$	$e6$	$e5$	0

(1.46)

```
> ChangeFrame(salg2):
> DGEnvironment[GSpace]([e1,e2,e3,e4], [e5,e6,e7], G, vectorlabels
  = [X], formlabels = [sigma]);
```

$$G\ Space\colon\ G \tag{1.47}$$

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4],
  2);
```

$$S := \left[\sigma 1 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 2 + \frac{1}{2}\,\sigma 2 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 1, \sigma 2 \otimes \sigma 2, \frac{1}{2}\,\sigma 2 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 2, \frac{1}{2}\,\sigma 2 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 2, \sigma 3 \otimes \sigma 3, \frac{1}{2}\,\sigma 3 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 3, \sigma 4 \otimes \sigma 4\right] \tag{1.48}$$

```
> g := InvariantGeometricObjectFields([X5, X6, X7], S);
```

$$g := \_C1\,\sigma 1 \otimes \sigma 1 + \_C1\,\sigma 2 \otimes \sigma 2 + \_C2\,\sigma 3 \otimes \sigma 3 - \_C1\,\sigma 4 \otimes \sigma 4 \tag{1.49}$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$10 \tag{1.50}$$

```
> C := CurvatureTensor(g);
```

$$C := -\frac{\_C1}{\_C2}\,X1 \otimes \sigma 2 \otimes \sigma 1 \otimes \sigma 2 + \frac{\_C1}{\_C2}\,X1 \otimes \sigma 2 \otimes \sigma 2 \otimes \sigma 1 - X1 \otimes \sigma 3 \otimes \sigma 1 \otimes \sigma 3 + X1 \otimes \sigma 3 \otimes \sigma 3 \otimes \sigma 1 + \frac{\_C1}{\_C2}\,X1 \otimes \sigma 4 \otimes \sigma 1 \otimes \sigma 4 - \frac{\_C1}{\_C2}\,X1 \otimes \sigma 4 \otimes \sigma 4 \otimes \sigma 1 + \frac{\_C1}{\_C2}\,X2 \otimes \sigma 1 \otimes \sigma 1 \otimes \sigma 2 - \frac{\_C1}{\_C2}\,X2 \otimes \sigma 1 \otimes \sigma 2 \otimes \sigma 1 - X2 \otimes \sigma 3 \otimes \sigma 2 \otimes \sigma 3 + X2 \otimes \sigma 3 \otimes \sigma 3 \otimes \sigma 2 + \frac{\_C1}{\_C2}\,X2 \otimes \sigma 4 \otimes \sigma 2 \otimes \sigma 4 - \frac{\_C1}{\_C2}\,X2 \otimes \sigma 4 \otimes \sigma 4 \otimes \sigma 2 + \frac{\_C1}{\_C2}\,X3 \otimes \sigma 1 \otimes \sigma 1 \otimes \sigma 3 - \frac{\_C1}{\_C2}\,X3 \otimes \sigma 1 \otimes \sigma 3 \otimes \sigma 1 + \frac{\_C1}{\_C2}\,X3 \otimes \sigma 2 \otimes \sigma 2 \otimes \sigma 3 - \frac{\_C1}{\_C2}\,X3 \otimes \sigma 2 \otimes \sigma 3 \otimes \sigma 2 + \frac{\_C1}{\_C2}\,X3 \otimes \sigma 4 \otimes \sigma 3 \otimes \sigma 4 - \frac{\_C1}{\_C2}\,X3 \otimes \sigma 4 \otimes \sigma 4 \otimes \sigma 3 + \frac{\_C1}{\_C2}\,X4 \otimes \sigma 1 \otimes \sigma 1 \otimes \sigma 4 - \frac{\_C1}{\_C2}\,X4 \otimes \sigma 1 \otimes \sigma 4 \otimes \sigma 1 + \frac{\_C1}{\_C2}\,X4 \tag{1.51}$$

$\otimes \sigma 2 \otimes \sigma 2 \otimes \sigma 4 - \frac{\_C1}{\_C2} X4 \otimes \sigma 2 \otimes \sigma 4 \otimes \sigma 2 + X4 \otimes \sigma 3 \otimes \sigma 3 \otimes \sigma 4 - X4 \otimes \sigma 3 \otimes \sigma 4 \otimes \sigma 3$

```
> CovariantDerivative(C, Christoffel(g) );
```

$$0\, X1 \otimes \sigma 1 \otimes \sigma 1 \otimes \sigma 1 \otimes \sigma 1 \quad (1.52)$$

For fun observe the sectional curvature is constant for arbitrary vectors:

```
> SectionalCurvature(g, C, a*X1 + b*X2 + c*X3 +d*X4, r*X1 + s*X2 + t*X3 + u*X4 );
```

$$-\frac{1}{\_C2} \quad (1.53)$$

Here is the Ricci scalar:

```
> RicciScalar(g);
```

$$-\frac{12}{\_C2} \quad (1.54)$$

Subcase 2: h4=0. Flat.

```
> ChangeFrame(alg1):
> LD1b := eval(LieAlgebraData( [e1-(1/2)*c2*e7, e2+(1/2)*c2*e6, e3, e4+(1/2)*c2*e5, e5, e6, e7], alg1b), h4=0);
```

$LD1b := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -e2, [e1, e6] = -e4, [e1, e7] = 0, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = e1, [e2, e6] = 0, [e2, e7] = -e4, [e3, e4] = 0, [e3, e5] = 0, [e3, e6] = 0, [e3, e7] = 0, [e4, e5] = 0, [e4, e6] = -e1, [e4, e7] = -e2, [e5, e6] = e7, [e5, e7] = -e6, [e6, e7] = -e5$ (1.55)

```
> DGsetup(LD1b):
> DGEnvironment[GSpace]([e1,e2,e3,e4], [e5,e6,e7], G, vectorlabels = [X], formlabels = [sigma]);
```

$$G\ Space\!: G \quad (1.56)$$

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4], 2);
```

$S := \left[\sigma 1 \otimes \sigma 1, \frac{1}{2} \sigma 1 \otimes \sigma 2 + \frac{1}{2} \sigma 2 \otimes \sigma 1, \frac{1}{2} \sigma 1 \otimes \sigma 3 + \frac{1}{2} \sigma 3 \otimes \sigma 1, \frac{1}{2} \sigma 1 \otimes \sigma 4 + \frac{1}{2} \sigma 4 \otimes \sigma 1, \sigma 2 \otimes \sigma 2, \frac{1}{2} \sigma 2 \otimes \sigma 3 + \frac{1}{2} \sigma 3 \otimes \sigma 2, \frac{1}{2} \sigma 2 \otimes \sigma 4 + \frac{1}{2} \sigma 4 \otimes \sigma 2, \sigma 3 \otimes \sigma 3, \frac{1}{2} \sigma 3 \otimes \sigma 4 + \frac{1}{2} \sigma 4 \otimes \sigma 3, \sigma 4 \otimes \sigma 4\right]$ (1.57)

```
> g := InvariantGeometricObjectFields([X5, X6, X7], S);
```

$$g := \_C1\, \sigma 1 \otimes \sigma 1 + \_C1\, \sigma 2 \otimes \sigma 2 + \_C2\, \sigma 3 \otimes \sigma 3 - \_C1\, \sigma 4 \otimes \sigma 4 \quad (1.58)$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$10 \quad (1.59)$$

```
> CurvatureTensor(g);
```

$$0\, X1 \otimes \sigma 1 \otimes \sigma 1 \otimes \sigma 1 \quad (1.60)$$

A.5.3 $F5$

Maple Worksheet
Seven-dimensional Lie algebra
Three-dimensional Isotropy
Isotropy Type F5

These Maple worksheets aim to validate the claims of chapter 3 regarding the Schmidt method.

F5: {B(θ),B3, B4}, note θϵ(0,π/2), - only a flat case

Note the definition of B(θ):

```
> B(theta) := cos(theta)*Rz - sin(theta)*Kz;
```

$$B(\theta) := \begin{bmatrix} 0 & -\cos(\theta) & 0 & 0 \\ \cos(\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta) \\ 0 & 0 & -\sin(\theta) & 0 \end{bmatrix} \quad (1.1)$$

Here is the basis of F5:

```
> B(theta), B3, B4;
```

$$\begin{bmatrix} 0 & -\cos(\theta) & 0 & 0 \\ \cos(\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta) \\ 0 & 0 & -\sin(\theta) & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad (1.2)$$

Observe the bracket relations of this subalgebra:

```
> e5e6 := B(theta).B3 - B3.B(theta); # = sin(theta)e6 + cos(theta)
  e7
```

$$e5e6 := \begin{bmatrix} 0 & 0 & -\sin(\theta) & -\sin(\theta) \\ 0 & 0 & -\cos(\theta) & -\cos(\theta) \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ -\sin(\theta) & -\cos(\theta) & 0 & 0 \end{bmatrix} \quad (1.3)$$

```
> e5e7 := B(theta).B4 - B4.B(theta); # = -cos(theta)e6 + sin(theta)
  e7
```

$$e5e7 := \begin{bmatrix} 0 & 0 & \cos(\theta) & \cos(\theta) \\ 0 & 0 & -\sin(\theta) & -\sin(\theta) \\ -\cos(\theta) & \sin(\theta) & 0 & 0 \\ \cos(\theta) & -\sin(\theta) & 0 & 0 \end{bmatrix} \quad (1.4)$$

```
> e6e7 := B3.B4 - B4.B3; # = 0
```

$$e6e7 := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.5}$$

By assuming the above matrices define the adjoint action of e5, e6, and e7 respectively, restricted to a reductive complement m = span({e1, e2, e3, e4}), we obtain the following structure equations:

```
> LDx := LieAlgebraData([
  '[e1,e2]= a1*e1+a2*e2+a3*e3+a4*e4+a5*e5+a6*e6+a7*e7',
  '[e1,e3]= b1*e1+b2*e2+b3*e3+b4*e4+b5*e5+b6*e6+b7*e7',
  '[e1,e4]= c1*e1+c2*e2+c3*e3+c4*e4+c5*e5+c6*e6+c7*e7',
  '[e1,e5]= -cos(theta)*e2',
  '[e1,e6]= -e3+e4',
  '[e1,e7]= 0',
  '[e2,e3]=e1*f1+e2*f2+e3*f3+e4*f4+e5*f5+e6*f6+e7*f7',
  '[e2,e4]=e1*g1+e2*g2+e3*g3+e4*g4+e5*g5+e6*g6+e7*g7',
  '[e2,e5]= cos(theta)*e1',
  '[e2,e6]= 0',
  '[e2,e7]= -e3+e4',

  '[e3,e4]=e1*h1+e2*h2+e3*h3+e4*h4+e5*h5+e6*h6+e7*h7',
  '[e3,e5]= sin(theta)*e4',
  '[e3,e6]= e1',
  '[e3,e7]= e2',
  '[e4,e5]= sin(theta)*e3',
  '[e4,e6]= e1',
  '[e4,e7]= e2',

  '[e5,e6]= sin(theta)*e6 + cos(theta)*e7',
  '[e5,e7]= -cos(theta)*e6 + sin(theta)*e7',

  '[e6,e7]= 0'],

  ['e1','e2','e3','e4','e5','e6', 'e7'],algx);
```

$$LDx := [e1, e2] = a1\ e1 + a2\ e2 + a3\ e3 + a4\ e4 + a5\ e5 + a6\ e6 + a7\ e7,\ [e1, e3] = b1\ e1 + b2\ e2 + b3\ e3 + b4\ e4 + b5\ e5 + b6\ e6 + b7\ e7,\ [e1, e4] = c1\ e1 + c2\ e2 + c3\ e3 + c4\ e4 + c5\ e5 + c6\ e6 + c7\ e7,\ [e1, e5] = -\cos(\theta)\ e2,\ [e1, e6] = -e3 + e4,\ [e1, e7] = 0,\ [e2, e3] = f1\ e1 + f2\ e2 + f3\ e3 + f4\ e4 + f5\ e5 + f6\ e6 + f7\ e7,\ [e2, e4] = g1\ e1 + g2\ e2 + g3\ e3 + g4\ e4 + g5\ e5 + g6\ e6 + g7\ e7,\ [e2, e5] = \cos(\theta)\ e1,\ [e2, e6] = 0,\ [e2, e7] = -e3 + e4,\ [e3, e4] = h1\ e1 + h2\ e2 + h3\ e3 + h4\ e4 + h5\ e5 + h6\ e6 + h7\ e7,\ [e3, e5] = \sin(\theta)\ e4,\ [e3, e6] = e1,\ [e3, e7] = e2,\ [e4, e5] = \sin(\theta)\ e3,\ [e4, e6] = e1,\ [e4, e7] = e2,\ [e5, e6] = \sin(\theta)\ e6 + \cos(\theta)\ e7,\ [e5, e7] = -\cos(\theta)\ e6 + \sin(\theta)\ e7,\ [e6, e7] = 0 \tag{1.6}$$

Initialize the Lie algebra:

```
> DGsetup(LDx, [e], [theta]):
```

Observe the adjoint representation of the isotropy
restricted to the reductive complement.

```
> Adjoint(e5, [e1,e2,e3,e4]), Adjoint(e6, [e1,e2,e3,e4]), Adjoint
  (e7, [e1,e2,e3,e4]);
```

$$\begin{bmatrix} 0 & -\cos(\theta) & 0 & 0 \\ \cos(\theta) & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sin(\theta) \\ 0 & 0 & -\sin(\theta) & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{1.7}$$

Extract the linear equations needing solving upon imposing
the Jacobi identities:

```
> ChangeFrame(algx):
  lineqs := []:
  for i from 1 to 7 do
  lineqs||i := []:
   cfs := convert(DGinformation(ExteriorDerivative
  (ExteriorDerivative(theta||i)), "CoefficientSet"), list):
   tf := map(type, cfs, linear):
   cnsts := map(type, cfs, constant):
   if nops(cnsts)>= 1 then
    for l from 1 to nops(cnsts) do
     if cnsts[l] then
      error cnsts[l], "Contradiction, there is a constant
  coefficient.";
     fi;
    od;
   fi;
   for k from 1 to nops(cfs) do
    if tf[k] then
     lineqs||i := [op(lineqs||i), cfs[k]]:
    fi:
   od:
  od:
  for j from 1 to 7 do
   lineqs := [op(lineqs), op(lineqs||j)]:
  od:
  lineqs := convert(lineqs, set):
```

Here are the linear equations:

```
> lineqs;
```

$$\{a2, a5, h1, h2, h5, -a1, -a5, a3 - f1, -a4 + c2, a4 + g1, -b1 + h3, b1 + h4, -b2 - a3, b2 - a4, -c1 + b1, -c1 + h3, c1 + h4, -c2 - a3, -c2 + b2, -c5 + b5, f1 + a4, -f2 + g2, -f2 + h3, f2 + h4, -f5 + g5, -g1 + a3, -g1 + f1, -g2 + h3, g2 + h4, -g5 + f5, -a1 - f3 + g3, a1 + f3 + f4, a1 - f4 + g4, a1 + g3 + g4, -a2 + b3 + b4, -a2 + b3 - c3, -a2 + c3 + c4, a2 + b4 - c4, b3 + b4 + h1, c3 + h1 + c4, f3 + f4 + h2, g3 + h2 + g4, -h1 - c3 + b3, h1 - c4 + b4, -h2 - g3 + f3, h2 - g4 + f4, a3 + b2 - c2 + a4, a3 - f1 + a4 + g1, -c1 + b1 + h3 + h4, -g2 + f2 + h3 + h4\} \tag{1.8}$$

Solve for the unknowns:

```
> sol := solve(lineqs);
```

(1.9)

$$sol := \{a1 = 0, a2 = 0, a3 = -a4, a4 = a4, a5 = 0, b1 = -h4, b2 = a4, b3 = -c4, b4 = c4, b5 = c5, c1 = -h4, c2 = a4, c3 = -c4, c4 = c4, c5 = c5, f1 = -a4, f2 = -h4, f3 = -g4, f4 = g4, f5 = g5, g1 = -a4, g2 = -h4, g3 = -g4, g4 = g4, g5 = g5, h1 = 0, h2 = 0, h3 = -h4, h4 = h4, h5 = 0\} \quad \textbf{(1.9)}$$

Substitute the solution into the Lie algebra:

```
> LDxx := eval(LDx, sol union {algx=algxx});
```

$$LDxx := [e1, e2] = -a4\, e3 + a4\, e4 + a6\, e6 + a7\, e7, [e1, e3] = -h4\, e1 + a4\, e2 - c4\, e3 + c4\, e4 + c5\, e5 + b6\, e6 + b7\, e7, [e1, e4] = -h4\, e1 + a4\, e2 - c4\, e3 + c4\, e4 + c5\, e5 + c6\, e6 + c7\, e7, [e1, e5] = -\cos(\theta)\, e2, [e1, e6] = -e3 + e4, [e1, e7] = 0, [e2, e3] = -a4\, e1 - h4\, e2 - g4\, e3 + g4\, e4 + g5\, e5 + f6\, e6 + f7\, e7, [e2, e4] = -a4\, e1 - h4\, e2 - g4\, e3 + g4\, e4 + g5\, e5 + g6\, e6 + g7\, e7, [e2, e5] = \cos(\theta)\, e1, [e2, e6] = 0, [e2, e7] = -e3 + e4, [e3, e4] = -h4\, e3 + h4\, e4 + h6\, e6 + h7\, e7, [e3, e5] = \sin(\theta)\, e4, [e3, e6] = e1, [e3, e7] = e2, [e4, e5] = \sin(\theta)\, e3, [e4, e6] = e1, [e4, e7] = e2, [e5, e6] = \sin(\theta)\, e6 + \cos(\theta)\, e7, [e5, e7] = -\cos(\theta)\, e6 + \sin(\theta)\, e7, [e6, e7] = 0 \quad \textbf{(1.10)}$$

```
> DGsetup(LDxx):
```

Here are the remaining unknowns:

```
> par := indets(LDxx) minus {LDxx, algxx} minus {cos(theta), sin
  (theta), theta};
```

$$par := \{a4, a6, a7, b6, b7, c4, c5, c6, c7, f6, f7, g4, g5, g6, g7, h4, h6, h7\} \quad \textbf{(1.11)}$$

Imposing the Jacobi equations gives conditions
on remainins unknowns:

```
> ddtheta1:=ExteriorDerivative(ExteriorDerivative(theta1));
  ddtheta2:=ExteriorDerivative(ExteriorDerivative(theta2));
  ddtheta3:=ExteriorDerivative(ExteriorDerivative(theta3));
  ddtheta4:=ExteriorDerivative(ExteriorDerivative(theta4));
  ddtheta5:=ExteriorDerivative(ExteriorDerivative(theta5));
  ddtheta6:=ExteriorDerivative(ExteriorDerivative(theta6));
  ddtheta7:=ExteriorDerivative(ExteriorDerivative(theta7));
```

$$ddtheta1 := -(-c5\cos(\theta) + a6)\,\theta1 \wedge \theta2 \wedge \theta3 - (-c5\cos(\theta) + a6)\,\theta1 \wedge \theta2 \wedge \theta4 - (-c6 + b6)\,\theta1 \wedge \theta3 \wedge \theta4 + h4\sin(\theta)\,\theta1 \wedge \theta3 \wedge \theta5 + h4\sin(\theta)\,\theta1 \wedge \theta4 \wedge \theta5 - (-g6 + f6)\,\theta2 \wedge \theta3 \wedge \theta4 + a4\sin(\theta)\,\theta2 \wedge \theta3 \wedge \theta5 + a4\sin(\theta)\,\theta2 \wedge \theta4 \wedge \theta5$$

$$ddtheta2 := -(-g5\cos(\theta) + a7)\,\theta1 \wedge \theta2 \wedge \theta3 - (-g5\cos(\theta) + a7)\,\theta1 \wedge \theta2 \wedge \theta4 - (-c7 + b7)\,\theta1 \wedge \theta3 \wedge \theta4 - a4\sin(\theta)\,\theta1 \wedge \theta3 \wedge \theta5 - a4\sin(\theta)\,\theta1 \wedge \theta4 \wedge \theta5 - (-g7 + f7)\,\theta2 \wedge \theta3 \wedge \theta4 + h4\sin(\theta)\,\theta2 \wedge \theta3 \wedge \theta5 + h4\sin(\theta)\,\theta2 \wedge \theta4 \wedge \theta5$$

$$ddtheta3 := -(a4\, h4 + b7 - f6)\,\theta1 \wedge \theta2 \wedge \theta3 - (a4\, h4 + c7 - g6)\,\theta1 \wedge \theta2 \wedge \theta4 + a4\sin(\theta)\,\theta1 \wedge \theta2 \wedge \theta5 - (c5\sin(\theta) - h6)\,\theta1 \wedge \theta3 \wedge \theta4 + (-\cos(\theta)\, g4 + 2\, c4\sin(\theta))\,\theta1 \wedge \theta3 \wedge \theta5 + (-\cos(\theta)\, g4 + 2\, c4\sin(\theta))\,\theta1 \wedge \theta4 \wedge \theta5 - (g5\sin(\theta) - h7)\,\theta2 \wedge \theta3 \wedge \theta4 + (\cos(\theta)\, c4 + 2\, g4\sin(\theta))\,\theta2 \wedge \theta3 \wedge \theta5 + (\cos(\theta)\, c4 + 2\, g4\sin(\theta))\,\theta2 \wedge \theta4 \wedge \theta5 + h4\sin(\theta)\,\theta3 \wedge \theta4 \wedge \theta5$$

$$ddtheta4 := (a4\, h4 + b7 - f6)\, \theta 1 \wedge \theta 2 \wedge \theta 3 + (a4\, h4 + c7 - g6)\, \theta 1 \wedge \theta 2 \wedge \theta 4 - a4 \sin(\theta)\, \theta 1 \wedge \theta 2 \wedge \theta 5 + (c5 \sin(\theta) - h6)\, \theta 1 \wedge \theta 3 \wedge \theta 4 - (-\cos(\theta)\, g4 + 2\, c4 \sin(\theta))\, \theta 1 \wedge \theta 3 \wedge \theta 5 - (-\cos(\theta)\, g4 + 2\, c4 \sin(\theta))\, \theta 1 \wedge \theta 4 \wedge \theta 5 + (g5 \sin(\theta) - h7)\, \theta 2 \wedge \theta 3 \wedge \theta 4 - (\cos(\theta)\, c4 + 2\, g4 \sin(\theta))\, \theta 2 \wedge \theta 3 \wedge \theta 5 - (\cos(\theta)\, c4 + 2\, g4 \sin(\theta))\, \theta 2 \wedge \theta 4 \wedge \theta 5 - h4 \sin(\theta)\, \theta 3 \wedge \theta 4 \wedge \theta 5$$

$$ddtheta5 := -(-g5 \cos(\theta) + c5 \sin(\theta))\, \theta 1 \wedge \theta 3 \wedge \theta 5 - (-g5 \cos(\theta) + c5 \sin(\theta))\, \theta 1 \wedge \theta 4 \wedge \theta 5 - (c5 \cos(\theta) + g5 \sin(\theta))\, \theta 2 \wedge \theta 3 \wedge \theta 5 - (c5 \cos(\theta) + g5 \sin(\theta))\, \theta 2 \wedge \theta 4 \wedge \theta 5$$

$$ddtheta6 := -(a4\, h6 - 2\, h4\, a6 - g4\, b6 + c4\, f6 - c4\, g6 + g4\, c6)\, \theta 1 \wedge \theta 2 \wedge \theta 3 - (a4\, h6 - 2\, h4\, a6 - g4\, b6 + c4\, f6 - c4\, g6 + g4\, c6)\, \theta 1 \wedge \theta 2 \wedge \theta 4 - (a6 \sin(\theta) - a7 \cos(\theta))\, \theta 1 \wedge \theta 2 \wedge \theta 5 - (-g6 + f6)\, \theta 1 \wedge \theta 2 \wedge \theta 6 + (-c6 + b6)\, \theta 1 \wedge \theta 2 \wedge \theta 7 - (a4\, f6 - a4\, g6 - 2\, h4\, b6 + 2\, h4\, c6)\, \theta 1 \wedge \theta 3 \wedge \theta 4 - (-\cos(\theta)\, f6 + \sin(\theta)\, c6 + \sin(\theta)\, b6 - b7 \cos(\theta))\, \theta 1 \wedge \theta 3 \wedge \theta 5 + (h6 + c5 \sin(\theta))\, \theta 1 \wedge \theta 3 \wedge \theta 6 - (a6 + c5 \cos(\theta))\, \theta 1 \wedge \theta 3 \wedge \theta 7 - (-\cos(\theta)\, g6 + \sin(\theta)\, b6 + \sin(\theta)\, c6 - c7 \cos(\theta))\, \theta 1 \wedge \theta 4 \wedge \theta 5 + (h6 + c5 \sin(\theta))\, \theta 1 \wedge \theta 4 \wedge \theta 6 - (a6 + c5 \cos(\theta))\, \theta 1 \wedge \theta 4 \wedge \theta 7 + (a4\, b6 - a4\, c6 + 2\, h4\, f6 - 2\, h4\, g6)\, \theta 2 \wedge \theta 3 \wedge \theta 4 - (\cos(\theta)\, b6 + \sin(\theta)\, g6 + \sin(\theta)\, f6 - f7 \cos(\theta))\, \theta 2 \wedge \theta 3 \wedge \theta 5 + (a6 + g5 \sin(\theta))\, \theta 2 \wedge \theta 3 \wedge \theta 6 - (-h6 + g5 \cos(\theta))\, \theta 2 \wedge \theta 3 \wedge \theta 7 - (\cos(\theta)\, c6 + \sin(\theta)\, f6 + \sin(\theta)\, g6 - g7 \cos(\theta))\, \theta 2 \wedge \theta 4 \wedge \theta 5 + (a6 + g5 \sin(\theta))\, \theta 2 \wedge \theta 4 \wedge \theta 6 - (-h6 + g5 \cos(\theta))\, \theta 2 \wedge \theta 4 \wedge \theta 7 - (h6 \sin(\theta) - h7 \cos(\theta))\, \theta 3 \wedge \theta 4 \wedge \theta 5 + (-c6 + b6)\, \theta 3 \wedge \theta 4 \wedge \theta 6 + (-g6 + f6)\, \theta 3 \wedge \theta 4 \wedge \theta 7$$

$$ddtheta7 := -(a4\, h7 - 2\, h4\, a7 - g4\, b7 + c4\, f7 - c4\, g7 + g4\, c7)\, \theta 1 \wedge \theta 2 \wedge \theta 3 - (a4\, h7 - 2\, h4\, a7 - g4\, b7 + c4\, f7 - c4\, g7 + g4\, c7)\, \theta 1 \wedge \theta 2 \wedge \theta 4 - (a6 \cos(\theta) + a7 \sin(\theta))\, \theta 1 \wedge \theta 2 \wedge \theta 5 - (-g7 + f7)\, \theta 1 \wedge \theta 2 \wedge \theta 6 + (-c7 + b7)\, \theta 1 \wedge \theta 2 \wedge \theta 7 - (a4\, f7 - a4\, g7 - 2\, h4\, b7 + 2\, h4\, c7)\, \theta 1 \wedge \theta 3 \wedge \theta 4 - (-f7 \cos(\theta) + \sin(\theta)\, c7 + \cos(\theta)\, b6 + \sin(\theta)\, b7)\, \theta 1 \wedge \theta 3 \wedge \theta 5 + (h7 + c5 \cos(\theta))\, \theta 1 \wedge \theta 3 \wedge \theta 6 - (a7 - c5 \sin(\theta))\, \theta 1 \wedge \theta 3 \wedge \theta 7 - (-g7 \cos(\theta) + \sin(\theta)\, b7 + \cos(\theta)\, c6 + \sin(\theta)\, c7)\, \theta 1 \wedge \theta 4 \wedge \theta 5 + (h7 + c5 \cos(\theta))\, \theta 1 \wedge \theta 4 \wedge \theta 6 - (a7 - c5 \sin(\theta))\, \theta 1 \wedge \theta 4 \wedge \theta 7 + (a4\, b7 - a4\, c7 + 2\, h4\, f7 - 2\, h4\, g7)\, \theta 2 \wedge \theta 3 \wedge \theta 4 - (b7 \cos(\theta) + \sin(\theta)\, g7 + \cos(\theta)\, f6 + \sin(\theta)\, f7)\, \theta 2 \wedge \theta 3 \wedge \theta 5 + (a7 + g5 \cos(\theta))\, \theta 2 \wedge \theta 3 \wedge \theta 6 + (h7 + g5 \sin(\theta))\, \theta 2 \wedge \theta 3 \wedge \theta 7 - (c7 \cos(\theta) + \sin(\theta)\, f7 + \cos(\theta)\, g6 + \sin(\theta)\, g7)\, \theta 2 \wedge \theta 4 \wedge \theta 5 + (a7 + g5 \cos(\theta))\, \theta 2 \wedge \theta 4 \wedge \theta 6 + (h7 + g5 \sin(\theta))\, \theta 2 \wedge \theta 4 \wedge \theta 7 - (h6 \cos(\theta) + h7 \sin(\theta))\, \theta 3 \wedge \theta 4 \wedge \theta 5 + (-c7 + b7)\, \theta 3 \wedge \theta 4 \wedge \theta 6 + (-g7 + f7)\, \theta 3 \wedge \theta 4 \wedge \theta 7 \qquad \textbf{(1.12)}$$

We extract all coefficients of the 3-forms above and setting equal to zero, solve:

```
> eq := map(op, map(DGinformation, {ddtheta1, ddtheta2, ddtheta3,
```

```
 ddtheta4, ddtheta5, ddtheta6,ddtheta7} , "CoefficientSet"));
```

$$eq := \{a4\sin(\theta), h4\sin(\theta), -a4\sin(\theta), -h4\sin(\theta), -a6 - c5\cos(\theta), a6 + g5\sin(\theta), -a7 + c5\sin(\theta), a7 + g5\cos(\theta), -b6 + c6, -b7 + c7, -c6 + b6, -c7 + b7, -f6 + g6, -f7 + g7, -g6 + f6, -g7 + f7, h6 + c5\sin(\theta), h6 - g5\cos(\theta), h7 + c5\cos(\theta), h7 + g5\sin(\theta), -a6\cos(\theta) - a7\sin(\theta), -a6\sin(\theta) + a7\cos(\theta), -c5\cos(\theta) - g5\sin(\theta), c5\cos(\theta) - a6, -c5\sin(\theta) + h6, c5\sin(\theta) - h6, g5\cos(\theta) - a7, g5\cos(\theta) - c5\sin(\theta), -g5\sin(\theta) + h7, g5\sin(\theta) - h7, -h6\cos(\theta) - h7\sin(\theta), -h6\sin(\theta) + h7\cos(\theta), -\cos(\theta)\, c4 - 2\, g4\sin(\theta), \cos(\theta)\, c4 + 2\, g4\sin(\theta), -\cos(\theta)\, g4 + 2\, c4\sin(\theta), \cos(\theta)\, g4 - 2\, c4\sin(\theta), -b7\cos(\theta) - \sin(\theta)\, g7 - \cos(\theta)\, f6 - \sin(\theta)\, f7, -c7\cos(\theta) - \sin(\theta)\, f7 - \cos(\theta)\, g6 - \sin(\theta)\, g7, f7\cos(\theta) - \sin(\theta)\, c7 - \cos(\theta)\, b6 - \sin(\theta)\, b7, g7\cos(\theta) - \sin(\theta)\, b7 - \cos(\theta)\, c6 - \sin(\theta)\, c7, -\cos(\theta)\, b6 - \sin(\theta)\, g6 - \sin(\theta)\, f6 + f7\cos(\theta), -\cos(\theta)\, c6 - \sin(\theta)\, f6 - \sin(\theta)\, g6 + g7\cos(\theta), \cos(\theta)\, f6 - \sin(\theta)\, c6 - \sin(\theta)\, b6 + b7\cos(\theta), \cos(\theta)\, g6 - \sin(\theta)\, b6 - \sin(\theta)\, c6 + c7\cos(\theta), -a4\, h4 - b7 + f6, a4\, h4 + b7 - f6, -a4\, h4 - c7 + g6, a4\, h4 + c7 - g6, -a4\, f6 + a4\, g6 + 2\, b6\, h4 - 2\, c6\, h4, a4\, b6 - a4\, c6 + 2\, f6\, h4 - 2\, g6\, h4, -a4\, f7 + a4\, g7 + 2\, b7\, h4 - 2\, c7\, h4, a4\, b7 - a4\, c7 + 2\, f7\, h4 - 2\, g7\, h4, -a4\, h6 + 2\, a6\, h4 + b6\, g4 - c4\, f6 + c4\, g6 - c6\, g4, -a4\, h7 + 2\, a7\, h4 + b7\, g4 - c4\, f7 + c4\, g7 - c7\, g4\} \tag{1.13}$$

Observe that for generic θ the only solution is all parameters vanishing:

```
> sol := solve(eq, par, explicit);
```

$$sol := \{a4 = 0, a6 = 0, a7 = 0, b6 = 0, b7 = 0, c4 = 0, c5 = 0, c6 = 0, c7 = 0, f6 = 0, f7 = 0, g4 = 0, g5 = 0, g6 = 0, g7 = 0, h4 = 0, h6 = 0, h7 = 0\} \tag{1.14}$$

The above steps are consolidated by the call to the single command Query. The call returns true if a solution is found, provides the solution, and returns the Lie algebra structure given the solution. Note we have exactly what Query returns:

```
> H := Query(algxx, par, "Jacobi");
```

$$H := true, \{a4\sin(\theta), h4\sin(\theta), -a4\sin(\theta), -h4\sin(\theta), -a6 - c5\cos(\theta), a6 + g5\sin(\theta), -a7 + c5\sin(\theta), a7 + g5\cos(\theta), -b6 + c6, -b7 + c7, -c6 + b6, -c7 + b7, -f6 + g6, -f7 + g7, -g6 + f6, -g7 + f7, h6 + c5\sin(\theta), h6 - g5\cos(\theta), h7 + c5\cos(\theta), h7 + g5\sin(\theta), -a6\cos(\theta) - a7\sin(\theta), -a6\sin(\theta) + a7\cos(\theta), -c5\cos(\theta) - g5\sin(\theta), c5\cos(\theta) - a6, -c5\sin(\theta) + h6, c5\sin(\theta) - h6, g5\cos(\theta) - a7, g5\cos(\theta) - c5\sin(\theta), -g5\sin(\theta) + h7, g5\sin(\theta) - h7, -h6\cos(\theta) - h7\sin(\theta), -h6\sin(\theta) + h7\cos(\theta), -\cos(\theta)\, c4 - 2\, g4\sin(\theta), \cos(\theta)\, c4 + 2\, g4\sin(\theta), -\cos(\theta)\, g4 + 2\, c4\sin(\theta), \cos(\theta)\, g4 - 2\, c4\sin(\theta), \tag{1.15}$$

$-b7 \cos(\theta) - \sin(\theta)\, g7 - \cos(\theta)\, f6 - \sin(\theta)\, f7, -c7 \cos(\theta) - \sin(\theta)\, f7 - \cos(\theta)\, g6 - \sin(\theta)\, g7, f7 \cos(\theta) - \sin(\theta)\, c7 - \cos(\theta)\, b6 - \sin(\theta)\, b7, g7 \cos(\theta) - \sin(\theta)\, b7 - \cos(\theta)\, c6 - \sin(\theta)\, c7, -\cos(\theta)\, b6 - \sin(\theta)\, g6 - \sin(\theta)\, f6 + f7 \cos(\theta), -\cos(\theta)\, c6 - \sin(\theta)\, f6 - \sin(\theta)\, g6 + g7 \cos(\theta), \cos(\theta)\, f6 - \sin(\theta)\, c6 - \sin(\theta)\, b6 + b7 \cos(\theta), \cos(\theta)\, g6 - \sin(\theta)\, b6 - \sin(\theta)\, c6 + c7 \cos(\theta), -a4\, h4 - b7 + f6, a4\, h4 + b7 - f6, -a4\, h4 - c7 + g6, a4\, h4 + c7 - g6, -a4\, f6 + a4\, g6 + 2\, b6\, h4 - 2\, c6\, h4, a4\, b6 - a4\, c6 + 2\, f6\, h4 - 2\, g6\, h4, -a4\, f7 + a4\, g7 + 2\, b7\, h4 - 2\, c7\, h4, a4\, b7 - a4\, c7 + 2\, f7\, h4 - 2\, g7\, h4, -a4\, h6 + 2\, a6\, h4 + b6\, g4 - c4\, f6 + c4\, g6 - c6\, g4, -a4\, h7 + 2\, a7\, h4 + b7\, g4 - c4\, f7 + c4\, g7 - c7\, g4\}, [\{a4 = 0, a6 = 0, a7 = 0, b6 = 0, b7 = 0, c4 = 0, c5 = 0, c6 = 0, c7 = 0, f6 = 0, f7 = 0, g4 = 0, g5 = 0, g6 = 0, g7 = 0, h4 = 0, h6 = 0, h7 = 0\}], [[e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -\cos(\theta)\, e2, [e1, e6] = -e3 + e4, [e1, e7] = 0, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = \cos(\theta)\, e1, [e2, e6] = 0, [e2, e7] = -e3 + e4, [e3, e4] = 0, [e3, e5] = \sin(\theta)\, e4, [e3, e6] = e1, [e3, e7] = e2, [e4, e5] = \sin(\theta)\, e3, [e4, e6] = e1, [e4, e7] = e2, [e5, e6] = \sin(\theta)\, e6 + \cos(\theta)\, e7, [e5, e7] = -\cos(\theta)\, e6 + \sin(\theta)\, e7, [e6, e7] = 0]$

We rename and initialize the solution Lie algebra:

```
algf > LD1 := eval(H[4][1], "algxx_1"=alg1);
```

$$LD1 := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = -\cos(\theta)\, e2, [e1, e6] = -e3 + e4, [e1, e7] = 0, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = \cos(\theta)\, e1, [e2, e6] = 0, [e2, e7] = -e3 + e4, [e3, e4] = 0, [e3, e5] = \sin(\theta)\, e4, [e3, e6] = e1, [e3, e7] = e2, [e4, e5] = \sin(\theta)\, e3, [e4, e6] = e1, [e4, e7] = e2, [e5, e6] = \sin(\theta)\, e6 + \cos(\theta)\, e7, [e5, e7] = -\cos(\theta)\, e6 + \sin(\theta)\, e7, [e6, e7] = 0 \tag{1.16}$$

```
> DGsetup(LD1):
```

We investigate the isometry dimension:

```
> DGEnvironment[GSpace]([e1,e2,e3,e4], [e5,e6,e7], G, vectorlabels
  = [X], formlabels = [sigma]);
```

$$G\ Space:\ G \tag{1.17}$$

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4],
  2);
```

$$S := \left[\sigma1 \otimes \sigma1, \frac{1}{2}\, \sigma1 \otimes \sigma2 + \frac{1}{2}\, \sigma2 \otimes \sigma1, \frac{1}{2}\, \sigma1 \otimes \sigma3 + \frac{1}{2}\, \sigma3 \otimes \sigma1, \frac{1}{2}\, \sigma1 \otimes \sigma4 + \frac{1}{2}\, \sigma4 \otimes \sigma1, \sigma2 \otimes \sigma2, \frac{1}{2}\, \sigma2 \otimes \sigma3 + \frac{1}{2}\, \sigma3 \otimes \sigma2, \frac{1}{2}\, \sigma2 \otimes \sigma4 + \frac{1}{2}\, \sigma4 \otimes \sigma2, \sigma3 \otimes \sigma3, \frac{1}{2}\, \sigma3 \otimes \sigma4 + \frac{1}{2}\, \sigma4 \otimes \sigma3, \sigma4 \otimes \sigma4\right] \tag{1.18}$$

```
> g := InvariantGeometricObjectFields([X5, X6, X7], S);
```

$$g := \_C1\, \sigma1 \otimes \sigma1 + \_C1\, \sigma2 \otimes \sigma2 + \_C1\, \sigma3 \otimes \sigma3 - \_C1\, \sigma4 \otimes \sigma4 \tag{1.19}$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$10 \tag{1.20}$$

```
> CurvatureTensor(g);
```

$$0\, X1 \otimes \sigma 1 \otimes \sigma 1 \otimes \sigma 1 \tag{1.21}$$

Thus this case of isotropy type F5 is not included in the classification.

A.5.4 $F6$

Maple Worksheet
Seven-dimensional Lie algebra
Three-dimensional Isotropy
Isotropy Type F6

These Maple worksheets aim to validate the claims of chapter 3 regarding the Schmidt method.

F6: {B1, B3, B4} - intractable

Here is a basis of subalgebra F6:

```
> 1/2*B1, B3, B4;
```

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{1.1}$$

Observe the bracket relations of this subalgebra:

```
> 1/2*B1.B3 - B3.(1/2*B1); # = e7
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{1.2}$$

```
> 1/2*B1.B4 - B4.(1/2*B1); # = -e6
```

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{1.3}$$

```
> B3.B4 - B4.B3; # = 0
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{1.4}$$

By assuming the above matrices define the adjoint action of e5, e6, and e7 respectively, restricted to a reductive complement m = span({e1, e2, e3, e4}), we obtain the following structure equations:

```
> LDx := LieAlgebraData([
  '[e1,e2]=a1*e1+a2*e2+a3*e3+a4*e4+a5*e5+a6*e6+a7*e7',
```

```
'[e1,e3]=b1*e1+b2*e2+b3*e3+b4*e4+b5*e5+b6*e6+b7*e7',
'[e1,e4]=c1*e1+c2*e2+c3*e3+c4*e4+c5*e5+c6*e6+c7*e7',
'[e1,e5]= -1*e2',
'[e1,e6]= -e3+e4',
'[e1,e7]= 0',
'[e2,e3]=e1*f1+e2*f2+e3*f3+e4*f4+e5*f5+e6*f6+e7*f7',
'[e2,e4]=e1*g1+e2*g2+e3*g3+e4*g4+e5*g5+e6*g6+e7*g7',
'[e2,e5]= 1*e1',
'[e2,e6]= 0',
'[e2,e7]= -e3+e4',

'[e3,e4]=e1*h1+e2*h2+e3*h3+e4*h4+e5*h5+e6*h6+e7*h7',
'[e3,e5]= 0',
'[e3,e6]= e1',
'[e3,e7]= e2',
'[e4,e5]= 0',
'[e4,e6]= e1',
'[e4,e7]= e2',

'[e5,e6]= 1*e7',
'[e5,e7]=-1*e6',

'[e6,e7]= 0'],

['e1','e2','e3','e4','e5','e6', 'e7'],algx);
```

$$LDx := [e1, e2] = a1\,e1 + a2\,e2 + a3\,e3 + a4\,e4 + a5\,e5 + a6\,e6 + a7\,e7,\ [e1, e3] = b1\,e1 + b2\,e2 + b3\,e3 + b4\,e4 + b5\,e5 + b6\,e6 + b7\,e7,\ [e1, e4] = c1\,e1 + c2\,e2 + c3\,e3 + c4\,e4 + c5\,e5 + c6\,e6 + c7\,e7,\ [e1, e5] = -e2,\ [e1, e6] = -e3 + e4,\ [e1, e7] = 0,\ [e2, e3] = f1\,e1 + f2\,e2 + f3\,e3 + f4\,e4 + f5\,e5 + f6\,e6 + f7\,e7,\ [e2, e4] = g1\,e1 + g2\,e2 + g3\,e3 + g4\,e4 + g5\,e5 + g6\,e6 + g7\,e7,\ [e2, e5] = e1,\ [e2, e6] = 0,\ [e2, e7] = -e3 + e4,\ [e3, e4] = h1\,e1 + h2\,e2 + h3\,e3 + h4\,e4 + h5\,e5 + h6\,e6 + h7\,e7,\ [e3, e5] = 0,\ [e3, e6] = e1,\ [e3, e7] = e2,\ [e4, e5] = 0,\ [e4, e6] = e1,\ [e4, e7] = e2,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] = 0 \tag{1.5}$$

Initialize the Lie algebra:

```
> DGsetup(LDx, [e], [theta]):
```

Observe the adjoint representation of the isotropy restricted to the reductive complement.

```
> Adjoint(e5, [e1,e2,e3,e4]), Adjoint(e6, [e1,e2,e3,e4]), Adjoint
  (e7, [e1,e2,e3,e4]);
```

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{1.6}$$

Extract the linear equations needing solving upon imposing the Jacobi identities:

```
> ChangeFrame(algx):
  lineqs := []:
  for i from 1 to 7 do
```

```
lineqs||i := []:
 cfs := convert(DGinformation(
ExteriorDerivative(ExteriorDerivative(theta||i)),
"CoefficientSet"), list):
 tf := map(type, cfs, linear):
 cnsts := map(type, cfs, constant):
 if nops(cnsts)>= 1 then
  for l from 1 to nops(cnsts) do
   if cnsts[l] then
    error cnsts[l], "Contradiction, there is a constant
coefficient.";
   fi;
  od;
 fi;
 for k from 1 to nops(cfs) do
  if tf[k] then
   lineqs||i := [op(lineqs||i), cfs[k]]:
  fi:
 od:
od:
for j from 1 to 7 do
 lineqs := [op(lineqs), op(lineqs||j)]:
od:
lineqs := convert(lineqs, set):
```

Here are the linear equations:

```
> lineqs;
```

$$\{a2, a5, a6, a7, f3, f4, f5, g3, g4, g5, h1, h2, h5, h6, h7, -a1, -a5, -a6, -a7, -b3, -b4, -b5, -c3, -c4, -c5, -h1, -h6, a3 - f1, -a4 + c2, a4 + g1, -a6 - b5, -a6 - c5, a7 + f5, a7 + g5, -b1 + f2, -b1 + h3, b1 + h4, -b2 - a3, b2 - a4, b2 + f1, -b6 + f7, -b7 - f6, -c1 + b1, -c1 + g2, -c1 + h3, c1 + h4, -c2 - a3, -c2 + b2, c2 + g1, -c5 + b5, -c6 + b6, -c6 + g7, -c7 + b7, -c7 - g6, -f1 - b2, f1 + a4, -f2 + g2, -f2 + h3, f2 + h4, -f5 + g5, -f6 + g6, f6 + b7, -g1 + a3, -g1 - c2, -g1 + f1, -g2 + h3, g2 + h4, -g5 + f5, g6 + c7, -g7 + f7, h6 - f5, h6 - g5, h7 + b5, h7 + c5, -a1 - f3 + g3, a1 + f3 + f4, a1 - f4 + g4, a1 + g3 + g4, -a2 + b3 + b4, -a2 + b3 - c3, -a2 + c3 + c4, a2 + b4 - c4, b3 + b4 + h1, c3 + h1 + c4, -c6 + b6 - a5, -c7 + b7 + h5, f3 + f4 + h2, -f7 + g7 + a5, g3 + h2 + g4, -g6 + f6 - h5, -h1 - c3 + b3, h1 - c4 + b4, -h2 - g3 + f3, h2 - g4 + f4, a3 + b2 - c2 + a4, a3 - f1 + a4 + g1, -c1 + b1 + h3 + h4, -g2 + f2 + h3 + h4\} \qquad (1.7)$$

Solve for the unknowns:

```
> sol := solve(lineqs);
```

$$sol := \{a1 = 0, a2 = 0, a3 = -b2, a4 = b2, a5 = 0, a6 = 0, a7 = 0, b1 = -h4, b2 = b2, b3 = 0, b4 = 0, b5 = 0, b6 = g7, b7 = -g6, c1 = -h4, c2 = b2, c3 = 0, c4 = 0, c5 = 0, c6 = g7, c7 = -g6, f1 = -b2, f2 = -h4, f3 = 0, f4 = 0, f5 = 0, f6 = g6, f7 = g7, g1 = -b2, g2 = -h4, g3 = 0, g4 = 0, g5 = 0, g6 = g6, g7 = g7, h1 = 0, h2 = 0, h3 = -h4, h4 = h4, h5 = 0, h6 = 0, h7 = 0\} \qquad (1.8)$$

Substitute the solution into the Lie algebra:

```
> LDxx := eval(LDx, sol union {algx=algxx});
```

$LDxx := [e1, e2] = -b2\, e3 + b2\, e4,\ [e1, e3] = -h4\, e1 + b2\, e2 + g7\, e6 - g6\, e7,\ [e1, e4] = -h4\, e1 + b2\, e2 + g7\, e6 - g6\, e7,\ [e1, e5] = -e2,\ [e1, e6] = -e3 + e4,\ [e1, e7] = 0,\ [e2, e3] = -b2\, e1 - h4\, e2 + g6\, e6 + g7\, e7,\ [e2, e4] = -b2\, e1 - h4\, e2 + g6\, e6 + g7\, e7,\ [e2, e5] = e1,\ [e2, e6] = 0,\ [e2, e7] = -e3 + e4,\ [e3, e4] = -h4\, e3 + h4\, e4,\ [e3, e5] = 0,\ [e3, e6] = e1,\ [e3, e7] = e2,\ [e4, e5] = 0,\ [e4, e6] = e1,\ [e4, e7] = e2,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] = 0$ **(1.9)**

```
> DGsetup(LDxx):
```

We don't display the multiplication table at this stage.
Here are the remaining unknowns:

```
> par := indets(LDxx) minus {LDxx, algxx};
```

$$par := \{b2, g6, g7, h4\} \quad \textbf{(1.10)}$$

We impose the Jacobi identities in an attempt
to get conditions on the unknowns remaining:

```
> ddtheta1:=ExteriorDerivative(ExteriorDerivative(theta1));
  ddtheta2:=ExteriorDerivative(ExteriorDerivative(theta2));
  ddtheta3:=ExteriorDerivative(ExteriorDerivative(theta3));
  ddtheta4:=ExteriorDerivative(ExteriorDerivative(theta4));
  ddtheta5:=ExteriorDerivative(ExteriorDerivative(theta5));
  ddtheta6:=ExteriorDerivative(ExteriorDerivative(theta6));
  ddtheta6:=ExteriorDerivative(ExteriorDerivative(theta7));
```

$$ddtheta1 := 0\, \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta2 := 0\, \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta3 := -(b2\, h4 - 2\, g6)\, \theta1 \wedge \theta2 \wedge \theta3 - (b2\, h4 - 2\, g6)\, \theta1 \wedge \theta2 \wedge \theta4$$

$$ddtheta4 := (b2\, h4 - 2\, g6)\, \theta1 \wedge \theta2 \wedge \theta3 + (b2\, h4 - 2\, g6)\, \theta1 \wedge \theta2 \wedge \theta4$$

$$ddtheta5 := 0\, \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta6 := 0\, \theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta6 := 0\, \theta1 \wedge \theta2 \wedge \theta3 \quad \textbf{(1.11)}$$

This agrees with Query:

```
> H := Query(algxx, par, "Jacobi");
```

$H := true,\ \{0, -b2\, h4 + 2\, g6, b2\, h4 - 2\, g6\},\ \left[\left\{b2 = b2, g6 = \frac{1}{2}\, b2\, h4, g7 = g7, h4 = h4\right\}\right],\ \left[[e1, e2] = -b2\, e3 + b2\, e4,\ [e1, e3] = -h4\, e1 + b2\, e2 + g7\, e6 - \frac{b2\, h4}{2}\, e7,\ [e1, e4] = -h4\, e1 + b2\, e2 + g7\, e6 - \frac{b2\, h4}{2}\, e7,\ [e1, e5] = -e2,\ [e1, e6] = -e3 + e4,\ [e1, e7] = 0,\ [e2, e3] = -b2\, e1 - h4\, e2 + \frac{b2\, h4}{2}\, e6 + g7\, e7,\ [e2, e4] = -b2\, e1 - h4\, e2 + \frac{b2\, h4}{2}\, e6 + g7\, e7,\ [e2, e5] = e1,\ [e2, e6] = 0,\ [e2, e7] = -e3 + e4,\ [e3, e4] = -h4\, e3 + h4\, e4,\ [e3, e5] = 0,\ [e3, e6] = e1,\ [e3, e7] = e2,\ [e4, e5] = 0,\ [e4, e6] = e1,\ [e4, e7] = e2,\ [e5, e6] = e7,\ [e5, e7] = -e6,\ [e6, e7] = 0\right]$ **(1.12)**

We rename and initialize the solution:

```
algf > LD1 := eval(H[4][1],"algxx_1"=alg1 );
```

$$LD1 := [e1, e2] = -b2\,e3 + b2\,e4, \left[e1, e3\right] = -h4\,e1 + b2\,e2 + g7\,e6 - \frac{b2\,h4}{2}\,e7, \left[e1, e4\right] = -h4\,e1 + b2\,e2 + g7\,e6 - \frac{b2\,h4}{2}\,e7, [e1, e5] = -e2, [e1, e6] = -e3 + e4, [e1, e7] = 0, \left[e2, e3\right] = -b2\,e1 - h4\,e2 + \frac{b2\,h4}{2}\,e6 + g7\,e7, \left[e2, e4\right] = -b2\,e1 - h4\,e2 + \frac{b2\,h4}{2}\,e6 + g7\,e7, [e2, e5] = e1, [e2, e6] = 0, [e2, e7] = -e3 + e4, [e3, e4] = -h4\,e3 + h4\,e4, [e3, e5] = 0, [e3, e6] = e1, [e3, e7] = e2, [e4, e5] = 0, [e4, e6] = e1, [e4, e7] = e2, [e5, e6] = e7, [e5, e7] = -e6, [e6, e7] = 0 \tag{1.13}$$

```
> DGsetup(LD1):
```

We verify the isometry dimension of this Lie algebra:

```
> DGEnvironment[GSpace]([e1,e2,e3,e4], [e5,e6,e7], G, vectorlabels
  = [X], formlabels = [sigma]);
```

$$G\ Space\!: G \tag{1.14}$$

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4],
  2);
```

$$S := \left[\sigma1 \otimes \sigma1, \frac{1}{2}\,\sigma1 \otimes \sigma2 + \frac{1}{2}\,\sigma2 \otimes \sigma1, \frac{1}{2}\,\sigma1 \otimes \sigma3 + \frac{1}{2}\,\sigma3 \otimes \sigma1, \frac{1}{2}\,\sigma1 \otimes \sigma4 + \frac{1}{2}\,\sigma4 \otimes \sigma1, \sigma2 \otimes \sigma2, \frac{1}{2}\,\sigma2 \otimes \sigma3 + \frac{1}{2}\,\sigma3 \otimes \sigma2, \frac{1}{2}\,\sigma2 \otimes \sigma4 + \frac{1}{2}\,\sigma4 \otimes \sigma2, \sigma3 \otimes \sigma3, \frac{1}{2}\,\sigma3 \otimes \sigma4 + \frac{1}{2}\,\sigma4 \otimes \sigma3, \sigma4 \otimes \sigma4\right] \tag{1.15}$$

```
> g := InvariantGeometricObjectFields([X5, X6, X7], S);
```

$$g := \_C1\,\sigma1 \otimes \sigma1 + \_C1\,\sigma2 \otimes \sigma2 + \_C2\,\sigma3 \otimes \sigma3 - \left(\_C1 - \_C2\right)\sigma3 \otimes \sigma4 - \left(\_C1 - \_C2\right)\sigma4 \otimes \sigma3 - \left(2\,\_C1 - \_C2\right)\sigma4 \otimes \sigma4 \tag{1.16}$$

```
> IsometryAlgebraData(g, [], output = [ "Dimension"], algx);
```

$$7 \tag{1.17}$$

For generic values of the parameters, the isometry dimension is indeed 7.

Observe the following change of basis shows that the nilradical is $n_{5,3}$ of Snobl.

```
> ChangeFrame(alg1):
> LD2 := LieAlgebraData([e3-e4, e3-e4+e6, -e2+e3-e4-e7, e1+e3-e4+
  b2*e7, e7, e3, e5], alg2);
```

$$LD2 := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = -h4\,e1, [e1, e7] = 0, [e2, e3] = 0, [e2, e4] = e1, [e2, e5] = 0, [e2, e6] = -(-1 + h4)\,e1 - e4 + b2\,e5, [e2, e7] = -e5, [e3, e4] = 0, [e3, e5] = e1, \left[e3, e6\right] = \left(-b2 + 1 + \frac{1}{2}\,b2\,h4\right. \tag{1.18}$$

$\Big) e1 - \frac{b2\ h4}{2}\ e2 - \big(h4 + 1\big)\ e3 + b2\ e4 - \big(b2^2 + g7 + h4 + 1\big)\ e5,\ [e3, e7] = 2\ e1$
$- e2 - e4 + b2\ e5,\ [e4, e5] = 0,\ \big[e4, e6\big] = -g7\ e1 + g7\ e2 - h4\ e4 + \frac{b2\ h4}{2}\ e5,$
$[e4, e7] = -(1 + b2)\ e1 + b2\ e2 + e3 + e5,\ [e5, e6] = -e1 + e3 + e5,\ [e5, e7] =$
$-e1 + e2,\ [e6, e7] = 0$

```
> DGsetup(LD2):
```

Here is the nilradical:

```
> Nil  := Nilradical(alg2);
```

$$Nil := [e1, e2, e3, e4, e5] \quad \textbf{(1.19)}$$

We initialize the nilradical to view its structure
equations, from it's seen it is the Lie algebra $n_{5,\ 3}$
of Snobl:

```
> LDn  := LieAlgebraData(Nil, nil);DGsetup(LDn):
```

$LDn := [e1, e2] = 0,\ [e1, e3] = 0,\ [e1, e4] = 0,\ [e1, e5] = 0,\ [e2, e3] = 0,\ [e2, e4] = e1,$ **(1.20)**
$[e2, e5] = 0,\ [e3, e4] = 0,\ [e3, e5] = e1,\ [e4, e5] = 0$

```
> MultiplicationTable(nil, "LieTable");
```

nil	e1	e2	e3	e4	e5
e1	0	0	0	0	0
e2	0	0	0	e1	0
e3	0	0	0	0	e1
e4	0	− e1	0	0	0
e5	0	0	− e1	0	0

(1.21)

We wish to track the isotropy subalgebra in this new basis.
The following defines a matrix giving change of basis above:

```
> ChangeFrame(alg2):
> GC1  := GetComponents([e3-e4, e3-e4+e6, -e2+e3-e4-e7, e1+e3-e4+b2*
  e7, e7, e3, e5], [e1,e2,e3,e4,e5,e6,e7]);
```

$GC1 := [[0, 0, 1, -1, 0, 0, 0], [0, 0, 1, -1, 0, 1, 0], [0, -1, 1, -1, 0, 0, -1], [1, 0, 1, -1, 0,$ **(1.22)**
$0, b2], [0, 0, 0, 0, 0, 0, 1], [0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 1, 0, 0]]$

```
> Aa  := convert(GC1, Matrix)^+;
```

$$Aa := \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & b2 & 1 & 0 & 0 \end{bmatrix} \quad \textbf{(1.23)}$$

Use Aa to define a linear transformation:

```
> psi := LinearTransformation(alg1, alg2, Aa^(-1));
```

$$\psi := e1 \to -e1 + e4 - b2\, e5,\ e2 \to e1 - e3 - e5,\ e3 \to e6,\ e4 \to -e1 + e6,\ e5 \to e7,\ e6 \to -e1 + e2,\ e7 \to e5 \tag{1.24}$$

Note that psi is a homomorphism, as a sanity check:

```
> Query(alg1, alg2, Aa^(-1), {b2}, "Homomorphism");
```

$$true, \{0\}, [\{b2 = b2\}], \left[\left[\begin{array}{ccccccc} -1 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -b2 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]\right] \tag{1.25}$$

```
> ChangeFrame(alg1):
```

Here is the isotropy in the new basis:

```
> isoa := ApplyLinearTransformation((psi), [e5,e6,e7]);
```

$$isoa := [e7, -e1 + e2, e5] \tag{1.26}$$

We now want a reductive complement to the isotropy.
Here is a general complement:

```
> cba := ComplementaryBasis(isoa, t);
```

$$cba := [-(-1 + t2)\, e1 + t2\, e2 + t3\, e5 + t1\, e7, -t5\, e1 + t5\, e2 + e3 + t6\, e5 + t4\, e7, -t8\, e1 + t8\, e2 + e4 + t9\, e5 + t7\, e7, -t11\, e1 + t11\, e2 + t12\, e5 + e6 + t10\, e7], \{t1, t10, t11, t12, t2, t3, t4, t5, t6, t7, t8, t9\} \tag{1.27}$$

A call to Query finds the t-values giving a reductive pair:

```
> Query(isoa, cba, "ReductivePair");
```

$$true, \{0, t1, t12, t2, t3, t4, t7, -t1, -t11, -t12, -t2, -t3, -t7, -t1 + t4, -t1 + t7, t1 - t4, t1 - t7, -t2 + t8, t2 - t8, -t4 - t2, t4 + t2, t7 + t3, -1 - t3 + t6, -t10 - t2 + t5, t10 + t2 - t5, -1 + t6 - t2 + t8, 1 - t6 + t2 - t8, -b2 + t10 + t3 - t9, -b2 - t9 - t2 + t5, b2 + t9 + t2 - t5, -t5 + b2 - t3 + t9, -t8 + 1 + t3 - t6\}, [\{t1 = 0, t10 = b2 + t9, t11 = 0, t12 = 0, t2 = 0, t3 = 0, t4 = 0, t5 = b2 + t9, t6 = 1, t7 = 0, t8 = 0, t9 = t9\}], [[[e7, -e1 + e2, e5], [e1, -(b2 + t9)\, e1 + (b2 + t9)\, e2 + e3 + e5, e4 + t9\, e5, e6 + (b2 + t9)\, e7]]] \tag{1.28}$$

The following is a reductive complement to the isotropy:

```
> compa := [e1, e3+e5, e4-b2*e5, e6];
```

$$compa := [e1, e3 + e5, -b2\, e5 + e4, e6] \tag{1.29}$$

Observer the adjoint representation of the isotropy restricted
to a reductive complement is no longer in standard form but
is still type F6:

```
> rep := map(Adjoint, isoa, compa);
```

$$rep := \left[\begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right] \tag{1.30}$$

```
> IsotropyType(rep);
```

"F6" **(1.31)**

This concludes the investigation of type F6.

A.5.5 $F7$

Maple Worksheet
Seven-dimensional Lie algebra
Three-dimensional Isotropy
Isotropy Type F7

These Maple worksheets aim to validate the claims of chapter 3 regarding the Schmidt method.

F7: {B2, B3, B4} - flat

Here is a basis for the subalgebra F7:

```
> B2, B3, B4;
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad (1.1)$$

Observe the bracket relations:

```
> B2.B3-B3.B2; # = 2*e6
```

$$\begin{bmatrix} 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix} \quad (1.2)$$

```
> B2.B4 - B4.B2; # = 2*e7
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -2 \\ 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \quad (1.3)$$

```
> B3.B4-B4.B3; # = 0
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (1.4)$$

By assuming the above matrices define the adjoint action of e5, e6, and e7 respectively, restricted to a reductive complement m = span({e1, e2, e3, e4}), we obtain the following structure equations:

```
> LDx := LieAlgebraData([
  '[e1,e2]=a1*e1+a2*e2+a3*e3+a4*e4+a5*e5+a6*e6+a7*e7',
```

```
'[e1,e3]=b1*e1+b2*e2+b3*e3+b4*e4+b5*e5+b6*e6+b7*e7',
'[e1,e4]= c1*e1+c2*e2+c3*e3+c4*e4+c5*e5+c6*e6+c7*e7',
'[e1,e5]= 0',
'[e1,e6]= -e3+e4',
'[e1,e7]= 0',
'[e2,e3]=e1*f1+e2*f2+e3*f3+e4*f4+e5*f5+e6*f6+e7*f7',
'[e2,e4]=e1*g1+e2*g2+e3*g3+e4*g4+e5*g5+e6*g6+e7*g7',
'[e2,e5]= 0',
'[e2,e6]= 0',
'[e2,e7]= -e3+e4',

'[e3,e4]=e1*h1+e2*h2+e3*h3+e4*h4+e5*h5+e6*h6+e7*h7',
'[e3,e5]= 2*e4',
'[e3,e6]= e1',
'[e3,e7]= e2',
'[e4,e5]= 2*e3',
'[e4,e6]= e1',
'[e4,e7]= e2',

'[e5,e6]= 2*e6',
'[e5,e7]= 2*e7',

'[e6,e7]= 0'],

['e1','e2','e3','e4','e5','e6', 'e7'],algx);
```

$$LDx := [e1, e2] = a1\,e1 + a2\,e2 + a3\,e3 + a4\,e4 + a5\,e5 + a6\,e6 + a7\,e7,\ [e1, e3] = b1\,e1 + b2\,e2 + b3\,e3 + b4\,e4 + b5\,e5 + b6\,e6 + b7\,e7,\ [e1, e4] = c1\,e1 + c2\,e2 + c3\,e3 + c4\,e4 + c5\,e5 + c6\,e6 + c7\,e7,\ [e1, e5] = 0,\ [e1, e6] = -e3 + e4,\ [e1, e7] = 0,\ [e2, e3] = f1\,e1 + f2\,e2 + f3\,e3 + f4\,e4 + f5\,e5 + f6\,e6 + f7\,e7,\ [e2, e4] = g1\,e1 + g2\,e2 + g3\,e3 + g4\,e4 + g5\,e5 + g6\,e6 + g7\,e7,\ [e2, e5] = 0,\ [e2, e6] = 0,\ [e2, e7] = -e3 + e4,\ [e3, e4] = h1\,e1 + h2\,e2 + h3\,e3 + h4\,e4 + h5\,e5 + h6\,e6 + h7\,e7,\ [e3, e5] = 2\,e4,\ [e3, e6] = e1,\ [e3, e7] = e2,\ [e4, e5] = 2\,e3,\ [e4, e6] = e1,\ [e4, e7] = e2,\ [e5, e6] = 2\,e6,\ [e5, e7] = 2\,e7,\ [e6, e7] = 0 \quad \textbf{(1.5)}$$

Initialize the Lie algebra:

```
> DGsetup(LDx, [e], [theta]);
```

$$\textit{Lie algebra: algx} \quad \textbf{(1.6)}$$

Observe the adjoint representation of the isotropy restricted to the reductive complement.

```
> Adjoint(e5, [e1,e2,e3,e4]), Adjoint(e6, [e1,e2,e3,e4]), Adjoint
  (e7, [e1,e2,e3,e4]);
```

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad \textbf{(1.7)}$$

Extract the linear equations needing solving upon imposing the Jacobi identities:

```
> ChangeFrame(algx):
```

```
lineqs := []:
for i from 1 to 7 do
lineqs||i := []:
 cfs := convert(DGinformation(
ExteriorDerivative(ExteriorDerivative(theta||i)),
 "CoefficientSet"), list):
 tf := map(type, cfs, linear):
 cnsts := map(type, cfs, constant):
 if nops(cnsts)>= 1 then
  for l from 1 to nops(cnsts) do
   if cnsts[l] then
    error cnsts[l], "Contradiction, there is a constant
coefficient.";
   fi;
  od;
 fi;
 for k from 1 to nops(cfs) do
  if tf[k] then
   lineqs||i := [op(lineqs||i), cfs[k]]:
  fi:
 od:
od:
for j from 1 to 7 do
 lineqs := [op(lineqs), op(lineqs||j)]:
od:
lineqs := convert(lineqs, set):
```

Here are the linear equations:

```
> lineqs;
```

$$\{a2, a5, a7, h1, h2, h5, h6, h7, -a1, 2\,a3, 2\,a4, -a5, -2\,a6, -a6, -2\,a7, -2\,b1, -2\,b2, -2\,b5, -2\,c1, -2\,c2, -2\,c5, -2\,f1, -2\,f2, -2\,f5, -2\,g1, -2\,g2, -2\,g5, 2\,h3, 2\,h4, -2\,h6, -2\,h7, a3 - f1, -a4 + c2, a4 + g1, a6 + 2\,f5, a6 + 2\,g5, -a7 + 2\,b5, -a7 + 2\,c5, -b1 + h3, b1 + h4, -b2 - a3, b2 - a4, -2\,b3 + 2\,c4, 2\,b3 - 2\,c4, 2\,b4 - 2\,c3, -c1 + b1, -c1 + h3, c1 + h4, -c2 - a3, -c2 + b2, 2\,c3 - 2\,b4, -c5 + b5, -2\,c6 - 2\,b6, -c6 + b6, -2\,c7 - 2\,b7, -c7 + b7, f1 + a4, -f2 + g2, -f2 + h3, f2 + h4, -2\,f3 + 2\,g4, 2\,f3 - 2\,g4, 2\,f4 - 2\,g3, -f5 + g5, -f7 + g7, -g1 + a3, -g1 + f1, -g2 + h3, g2 + h4, 2\,g3 - 2\,f4, -g5 + f5, -2\,g6 - 2\,f6, -g6 + f6, -2\,g7 - 2\,f7, h6 + 2\,b5, h6 + 2\,c5, h7 + 2\,f5, h7 + 2\,g5, -a1 - f3 + g3, a1 + f3 + f4, a1 - f4 + g4, a1 + g3 + g4, -a2 + b3 + b4, -a2 + b3 - c3, -a2 + c3 + c4, a2 + b4 - c4, b3 + b4 + h1, c3 + h1 + c4, -c6 + b6 + 2\,h5, -c7 + b7 + 2\,a5, f3 + f4 + h2, -f6 + g6 + 2\,a5, g3 + h2 + g4, -g7 + f7 + 2\,h5, -h1 - c3 + b3, h1 - c4 + b4, -h2 - g3 + f3, h2 - g4 + f4, a3 + b2 - c2 + a4, a3 - f1 + a4 + g1, -c1 + b1 + h3 + h4, -g2 + f2 + h3 + h4\} \tag{1.8}$$

```
> sol := solve(lineqs);
```

$$sol := \{a1 = 0, a2 = 0, a3 = 0, a4 = 0, a5 = 0, a6 = 0, a7 = 0, b1 = 0, b2 = 0, b3 = 0, b4 = 0, b5 = 0, b6 = 0, b7 = 0, c1 = 0, c2 = 0, c3 = 0, c4 = 0, c5 = 0, c6 = 0, c7 = 0, f1 = 0, f2 = 0, f3 = 0, f4 = 0, f5 = 0, f6 = 0, f7 = 0, g1 = 0, g2 = 0, g3 = 0, g4 = 0, g5 = 0, g6 = 0, g7 = 0, h1 = 0, h2 = 0, h3 = 0, h4 = 0, h5 = 0, h6 = 0, h7 = 0\} \tag{1.9}$$

Substitute the solution into the Lie algebra:

```
> LDxx := eval(LDx, sol union {algx=algxx});
```

$$LDxx := [e1, e2] = 0, [e1, e3] = 0, [e1, e4] = 0, [e1, e5] = 0, [e1, e6] = -e3 + e4, [e1, e7] = 0, [e2, e3] = 0, [e2, e4] = 0, [e2, e5] = 0, [e2, e6] = 0, [e2, e7] = -e3 + e4, [e3, e4] = 0, [e3, e5] = 2\,e4, [e3, e6] = e1, [e3, e7] = e2, [e4, e5] = 2\,e3, [e4, e6] = e1, [e4, e7] = e2, [e5, e6] = 2\,e6, [e5, e7] = 2\,e7, [e6, e7] = 0 \tag{1.10}$$

```
> DGsetup(LDxx):
> MultiplicationTable(algxx, "LieTable");
```

algxx	$e1$	$e2$	$e3$	$e4$	$e5$	$e6$	$e7$
$e1$	0	0	0	0	0	$-e3+e4$	0
$e2$	0	0	0	0	0	0	$-e3+e4$
$e3$	0	0	0	0	$2\,e4$	$e1$	$e2$
$e4$	0	0	0	0	$2\,e3$	$e1$	$e2$
$e5$	0	0	$-2\,e4$	$-2\,e3$	0	$2\,e6$	$2\,e7$
$e6$	$e3-e4$	0	$-e1$	$-e1$	$-2\,e6$	0	0
$e7$	0	$e3-e4$	$-e2$	$-e2$	$-2\,e7$	0	0

(1.11)

As seen in the table, there are no remaining unknowns:

```
> par := indets(LDxx) minus {LDxx, algxx};
```

$$par := \{\ \} \tag{1.12}$$

As a sanity check we verify the Jacobi identities:

```
> ddtheta1:=ExteriorDerivative(ExteriorDerivative(theta1));
  ddtheta2:=ExteriorDerivative(ExteriorDerivative(theta2));
  ddtheta3:=ExteriorDerivative(ExteriorDerivative(theta3));
  ddtheta4:=ExteriorDerivative(ExteriorDerivative(theta4));
  ddtheta5:=ExteriorDerivative(ExteriorDerivative(theta5));
  ddtheta6:=ExteriorDerivative(ExteriorDerivative(theta6));
  ddtheta6:=ExteriorDerivative(ExteriorDerivative(theta7));
```

$$ddtheta1 := 0\,\theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta2 := 0\,\theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta3 := 0\,\theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta4 := 0\,\theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta5 := 0\,\theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta6 := 0\,\theta1 \wedge \theta2 \wedge \theta3$$

$$ddtheta6 := 0\,\theta1 \wedge \theta2 \wedge \theta3 \tag{1.13}$$

We investigate the isometry dimension. We construct the general metric on the homogeneous space and compute the dimension of its isometry algebra:

```
> DGEnvironment[GSpace]([e1,e2,e3,e4], [e5,e6,e7], G, vectorlabels
  = [X], formlabels = [sigma]);
```

$$G\ Space:\ G \tag{1.14}$$

```
> S := GenerateSymmetricTensors([sigma1, sigma2, sigma3, sigma4],
  2);
```

$$S := \left[\sigma 1 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 2 + \frac{1}{2}\,\sigma 2 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 1, \frac{1}{2}\,\sigma 1 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 1, \sigma 2 \otimes \sigma 2, \frac{1}{2}\,\sigma 2 \otimes \sigma 3 + \frac{1}{2}\,\sigma 3 \otimes \sigma 2, \frac{1}{2}\,\sigma 2 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 2, \sigma 3 \otimes \sigma 3, \frac{1}{2}\,\sigma 3 \otimes \sigma 4 + \frac{1}{2}\,\sigma 4 \otimes \sigma 3, \sigma 4 \otimes \sigma 4\right] \tag{1.15}$$

```
> g := InvariantGeometricObjectFields([X5, X6, X7], S);
```

$$g := \_C1\,\sigma 1 \otimes \sigma 1 + \_C1\,\sigma 2 \otimes \sigma 2 + \_C1\,\sigma 3 \otimes \sigma 3 - \_C1\,\sigma 4 \otimes \sigma 4 \tag{1.16}$$

```
> IsometryAlgebraData(g, [], output = ["Dimension"]);
```

$$10 \tag{1.17}$$

Observe the curvature tensor vanishes:

```
> CurvatureTensor(g);
```

$$0\,X1 \otimes \sigma 1 \otimes \sigma 1 \otimes \sigma 1 \tag{1.18}$$

Therefore this case is excluded. This concludes the investigation for type F7.

VITA

Jesse W. Hicks

Department of Mathematics and Statistics

Utah State University

Logan, UT 84322

j.hicks@aggiemail.usu.edu

Education

PhD, Mathematics, Utah State University, Logan, UT, 2016

Dissertation Title: "Classification of Spacetimes with Symmetry"

— Advisor: Prof. Ian Anderson

MS, Mathematics, Utah State University, Logan, UT 2011

Thesis Title: "Algebraic Properties of Killing Vectors For Lorentz Metrics in Four Dimensions"

— Advisor: Prof. Ian Anderson

BS, Mathematics, Southern Utah University, Cedar City, UT 2009

Computer Science Minor

Research Interest

Differential Geometry General Relativity Lie Theory

specifically, the classification of geometric structures such as Lorentz metrics in four and five dimensional relativity, as well as Bach tensors, symplectic geometries, and complex structures

Research Awards

Masters Graduate Researcher of the Year 2011-2012

Utah State University Department of Mathematics and Statistics

Teaching Awards

Bill E. Robins Award Nominee 2015-2016

Utah State University Graduate Teacher of the Year

College of Science Graduate Teacher of the Year 2015-2016

Utah State University College of Science

Selected from 160 other graduate students in the College of Science

Graduate Teacher of the Year 2015-2016

Utah State University Department of Mathematics and Statistics

Excellence In Teaching Award 2014-2015

Utah State University Department of Mathematics and Statistics

Teaching

Developed curriculum in all areas including syllabus, instruction, test/quiz preparation, holding office hours, student assessment for the following courses

Linear Algebra	Fall 2013
Calculus III	Summer 2013, Fall 2014
Calculus II	Fall 2012, Spring 2013, Summer 2014, Spring 2016
Calculus I	Fall 2011, Spring 2012, Summer 2012, Spring 2014, Spring 2015, Fall 2015
Calculus Techniques “Business Calculus”	Summer 2011
College Algebra	Fall 2009, Spring 2010 (Recitations only)
Elements of Algebra (precedes College Algebra)	Summer 2010, Fall 2010, Spring 2011

Presentations

Identification of Lorentzian Lie algebra-subalgebra Pairs in a Computer Algebra System
Joint Meeting of the Intermountain and Rocky Mountain Sections of the Mathematical Association of America
Colorado Mesa University, April 2016

Equivalence of Lie Algebra-Subalgebra Pairs and Why It Matters Geometrically
Graduate Student Conference in Algebra, Geometry, and Topology
Temple University, May 2015

Petrov's classification of spacetimes with symmetry and the Gödel metric
Graduate Research Symposium
Utah State University, April 2015

The Riemann curvature tensor, it's invariants, and their use in the classification of spacetimes
MAA Conference, Inter-mountain chapter
Brigham Young University, March 2015

Equivalence Problems In General Relativity: A Symmetry Approach
Graduate Student Symposium
Utah State University, April 2014

Algebraic Properties Of Killing Vectors For Lorentz Metrics In Four Dimensions
MAA Conference, Inter-mountain chapter
Utah Valley University, March 2014